

TRIGONOMETRIC APPLICATIONS

An ocean is a vast expanse that can be life-threatening to a person who experiences a disaster while boating. In order for help to arrive on time, it is necessary that the coast guard or a ship in the area be able to make an exact identification of the location. A distress signal sent out by the person in trouble can be analyzed by those receiving the signal from different directions. In this chapter we will derive the formulas that can be used to determine distances and angle measures when sufficient information is available.

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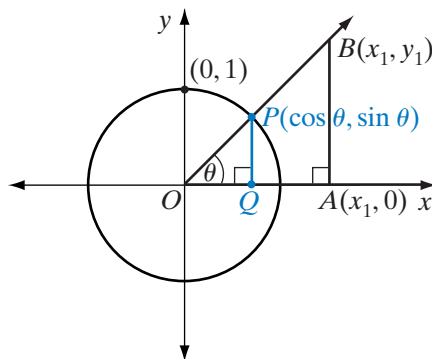
14-1 SIMILAR TRIANGLES

In the study of geometry, we learned that if two triangles are similar, the corresponding angles are congruent and the corresponding sides are in proportion. If certain pairs of corresponding angles and sides are congruent or proportional, then the triangles must be similar. The following pairs of congruent corresponding angles and proportional corresponding sides are sufficient to prove triangles similar.

1. Two angles (AA~)
2. Three sides (SSS~)
3. Two sides and the included angles (SAS~)
4. Hypotenuse and one leg of a right triangle (HL~)

We can use similar triangles to write the coordinates of a point in terms of its distance from the origin and the sine and cosine of an angle in standard position.

Let $B(x_1, y_1)$ be a point in the first quadrant of the coordinate plane with $OB = b$ and θ the measure of the angle formed by \overrightarrow{OB} and the positive ray of the x -axis. Let $P(\cos \theta, \sin \theta)$ be the point at which \overrightarrow{OB} intersects the unit circle. Let $Q(\cos \theta, 0)$ be the point at which a perpendicular line from P intersects the x -axis and $A(x_1, 0)$ be the point at which a perpendicular line from B intersects the x -axis. Then $\triangle OPQ \sim \triangle OBA$ by AA~. Therefore:



$$\frac{OA}{OQ} = \frac{OB}{OP}$$

$$\frac{|x_1 - 0|}{\cos \theta} = \frac{b}{1}$$

$$\frac{x_1}{\cos \theta} = \frac{b}{1}$$

$$x_1 = b \cos \theta$$

$$\frac{AB}{QP} = \frac{OB}{OP}$$

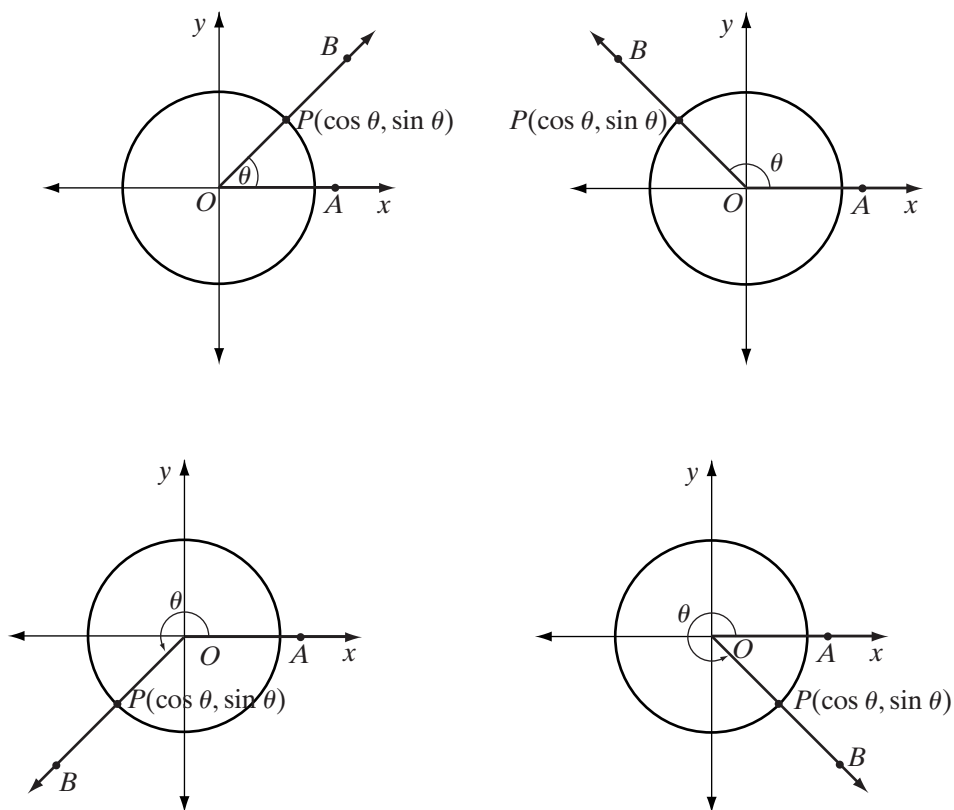
$$\frac{|y_1 - 0|}{\sin \theta} = \frac{b}{1}$$

$$\frac{y_1}{\sin \theta} = \frac{b}{1}$$

$$y_1 = b \sin \theta$$

Therefore, if B is a point b units from the origin of the coordinate plane and \overrightarrow{OB} is the terminal side of an angle in standard position whose measure is θ , then the coordinates of B are $(b \cos \theta, b \sin \theta)$. This statement can be shown to be true for any point in the coordinate plane.

Let $\angle AOB$ be an angle in standard position whose measure is θ . If the terminal side, \overrightarrow{OB} , intersects the unit circle at P , then the coordinates of P are $(\cos \theta, \sin \theta)$. The diagrams below show $\angle AOB$ in each quadrant.



A dilation of b with center at the origin will stretch each segment whose endpoint is the origin by a factor of b . Under the dilation D_b , the image of (x, y) is (bx, by) . If B is the image of $P(\cos \theta, \sin \theta)$ under the dilation D_b , then the coordinates of B are $(b \cos \theta, b \sin \theta)$. Therefore, $(b \cos \theta, b \sin \theta)$ are the coordinates of a point b units from the origin on the terminal ray of an angle in standard position whose measure is θ .

Any triangle can be positioned on the coordinate plane so that each vertex is identified by the coordinates of a point in the plane. The coordinates of the vertices can be expressed in terms of the trigonometric function values of angles in standard position.

EXAMPLE 1

Point S is 12 units from the origin and \overrightarrow{OS} makes an angle of 135° with the positive ray of the x -axis. What are the exact coordinates of S ?

Solution The coordinates of S are $(12 \cos 135^\circ, 12 \sin 135^\circ)$.

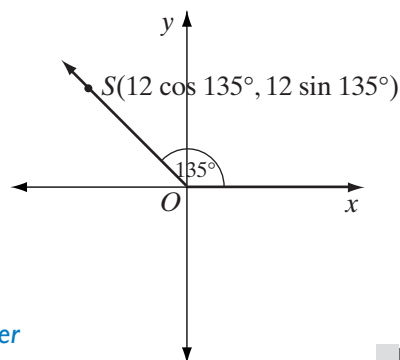
Since $135^\circ = 180^\circ - 45^\circ$,

$$\cos 135^\circ = -\cos 45^\circ = -\frac{\sqrt{2}}{2} \text{ and}$$

$$\sin 135^\circ = \sin 45^\circ = \frac{\sqrt{2}}{2}.$$

Therefore, the coordinates of S are

$$\left(12 \times \frac{-\sqrt{2}}{2}, 12 \times \frac{\sqrt{2}}{2}\right) = (-6\sqrt{2}, 6\sqrt{2}). \text{ Answer}$$



EXAMPLE 2

The coordinates of A are $(-5.30, -8.48)$.

a. Find OA to the nearest hundredth.

b. Find, to the nearest degree, the measure of the angle in standard position whose terminal side is \overrightarrow{OA} .

Solution a. Let $C(-5.30, 0)$ be the point at which a vertical line from A intersects the x -axis.

Then \overline{OA} is the hypotenuse of right $\triangle OAC$,

$$OC = |-5.30 - 0| = 5.30 \text{ and}$$

$$AC = |-8.48 - 0| = 8.48.$$

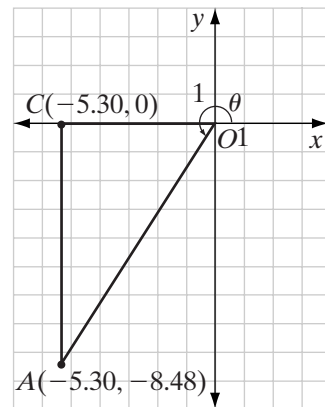
Using the Pythagorean Theorem:

$$OA^2 = OC^2 + AC^2$$

$$OA^2 = 5.30^2 + 8.48^2$$

$$OA = \sqrt{5.30^2 + 8.48^2}$$

We reject the negative root.



ENTER: **2nd** $\sqrt{}$ 5.30 x^2
 $+$ 8.48 x^2 $)$
ENTER

DISPLAY: $\sqrt{5.30^2 + 8.48^2}$
 10.00002

Write the measure of OA to the nearest hundredth: $OA = 10.00$.

- b. Let θ be the measure of the angle in standard position with terminal side \overrightarrow{OA} . A is in the third quadrant. We can use either coordinate to find the measure of θ , a third-quadrant angle.

$$(OA \cos \theta, OA \sin \theta) = (-5.30, -8.48)$$

$$10.00 \cos \theta = -5.30$$

$$\cos \theta = -0.530$$

ENTER: **2nd** **COS⁻¹** **(-)** 0.530
) **ENTER**

DISPLAY: $\cos^{-1}(-0.530)$
122.0054548

The calculator returns the measure of a second-quadrant angle. Use this measure to find the reference angle to the nearest degree.

$$R = 180 - 122 = 58^\circ$$

$$10.00 \sin \theta = -8.48$$

$$\sin \theta = -0.848$$

ENTER: **2nd** **SIN⁻¹** **(-)** 0.848
) **ENTER**

DISPLAY: $\sin^{-1}(-0.848)$
-57.99480014

The calculator returns the measure of a fourth-quadrant angle. Use this measure to find the reference angle to the nearest degree.

$$R = -(-58^\circ) = 58^\circ$$

Use the reference angle to find the measure of the third-quadrant angle.

$$\theta = 180 + 58 = 238^\circ$$

Answers a. $OA = 10.00$ b. $\theta = 238^\circ$

Exercises

Writing About Mathematics

1. In Example 2, is it possible to find the measure of θ without first finding OA ? Justify your answer.
2. In what quadrant is a point whose coordinates in radian measure are $(2 \cos \frac{4\pi}{3}, 2 \sin \frac{4\pi}{3})$? Justify your answer.

Developing Skills

In 3–14, write in simplest radical form the coordinates of each point A if A is on the terminal side of an angle in standard position whose degree measure is θ .

3. $OA = 4, \theta = 45^\circ$

4. $OA = 2, \theta = 30^\circ$

5. $OA = 6, \theta = 90^\circ$

6. $OA = 8, \theta = 120^\circ$

7. $OA = 15, \theta = 135^\circ$

8. $OA = 0.5, \theta = 180^\circ$

9. $OA = 9, \theta = 150^\circ$

10. $OA = 25, \theta = 210^\circ$

11. $OA = 12, \theta = 270^\circ$

12. $OA = \sqrt{2}, \theta = 225^\circ$

13. $OA = \sqrt{3}, \theta = 300^\circ$

14. $OA = 2, \theta = -60^\circ$

In 15–23, the coordinates of a point are given. **a.** Find the distance of the point from the origin. Express approximate distances to the nearest hundredth. **b.** Find the measure, to the nearest degree, of the angle in standard position whose terminal side contains the given point.

15. $(6, 8)$

16. $(-5, 12)$

17. $(0, 7)$

18. $(12, -9)$

19. $(15, 0)$

20. $(-8, -12)$

21. $(24, 7)$

22. $(6, -10)$

23. $(-8, 8)$

In 24–29, for each $\triangle ORS$, O is the origin, R is on the positive ray of the x -axis and \overline{PS} is the altitude from S to \overleftrightarrow{OR} . **a.** Find the exact coordinates of R and S . **b.** Find the exact area of $\triangle ORS$.

24. $OR = 5, m\angle ROS = \frac{\pi}{3}, OS = 3$

25. $OR = 12, m\angle ROS = \frac{\pi}{2}, OS = 8$

26. $OR = 8, m\angle ROS = \frac{3\pi}{4}, OS = 8$

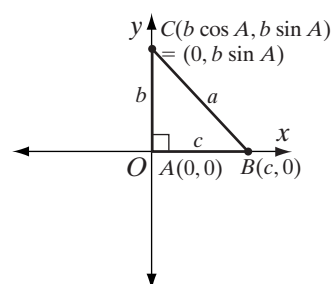
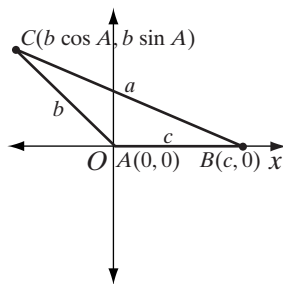
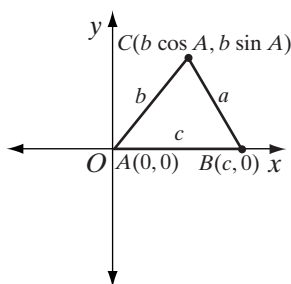
27. $OR = 20, OS = RS, PS = 10$

28. $OR = 9, \triangle ORS$ is equilateral

29. $OR = 7, m\angle ROS = \frac{\pi}{6}, PS = 8$

14-2 LAW OF COSINES

When we know the measures of two sides and the included angle of a triangle (SAS), the size and shape of the triangle are determined. Therefore, we should be able to find the measure of the third side of the triangle. In order to derive a formula to do this, we will position the triangle in the coordinate plane with one endpoint at the origin and one angle in standard position. As shown in the diagrams, the angle in standard position can be either acute, obtuse, or right.



Let $\triangle ABC$ be a triangle with $AB = c$, $BC = a$, and $CA = b$. The coordinates of A are $(0, 0)$, of B are $(c, 0)$, and of C are $(b \cos A, b \sin A)$. If b , c , and $m\angle A$ are known measures, then the coordinates of each vertex are known. We can find a , the length of the third side of the triangle, by using the distance formula.

The distance between the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by the formula:

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Let $P(x_1, y_1) = B(c, 0)$ and $Q(x_2, y_2) = C(b \cos A, b \sin A)$.

$$\begin{aligned} BC^2 &= (b \cos A - c)^2 + (b \sin A - 0)^2 \\ &= b^2 \cos^2 A - 2bc \cos A + c^2 + b^2 \sin^2 A \\ &= b^2 \cos^2 A + b^2 \sin^2 A + c^2 - 2bc \cos A \\ &= b^2 (\cos^2 A + \sin^2 A) + c^2 - 2bc \cos A \\ &= b^2(1) + c^2 - 2bc \cos A \\ &= b^2 + c^2 - 2bc \cos A \end{aligned}$$

If we let $BC = a$, we can write:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

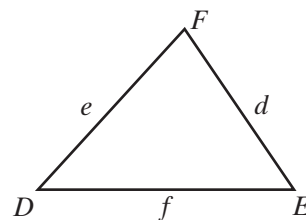
This formula is called the **Law of Cosines**. The law of cosines for $\triangle ABC$ can be written in terms of the measures of any two sides and the included angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

We can rewrite the Law of Cosines in terms of the letters that represent the vertices of any triangle. For example, in $\triangle DEF$, side \overline{DE} is opposite $\angle F$ so we let $DE = f$, side \overline{EF} is opposite $\angle D$ so we let $EF = d$, and side \overline{DF} is opposite $\angle E$ so we let $DF = e$. We can use the Law of Cosines to write a formula for the square of the measure of each side of $\triangle DEF$ in terms of the measures of the other two sides and the included angle.



$$d^2 = e^2 + f^2 - 2ef \cos D$$

$$e^2 = d^2 + f^2 - 2df \cos E$$

$$f^2 = d^2 + e^2 - 2de \cos F$$

EXAMPLE 1

In $\triangle ABC$, $AB = 8$, $AC = 10$, and $\cos A = \frac{1}{8}$. Find BC .

Solution $AB = c = 8$, $AC = b = 10$, and $\cos A = \frac{1}{8}$.

Use the Law of Cosines to find $BC = a$.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$a^2 = 10^2 + 8^2 - 2(10)(8)\left(\frac{1}{8}\right)$$

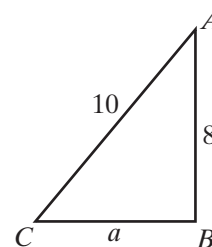
$$a^2 = 100 + 64 - 20$$

$$a^2 = 144$$

$$a = \pm 12$$

Since a is the length of a line segment, a is a positive number.

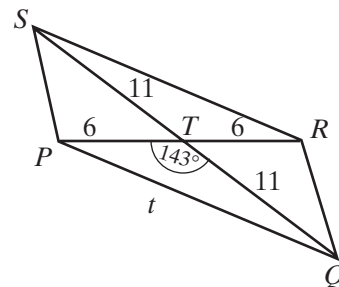
Answer $BC = 12$



EXAMPLE 2

The diagonals of a parallelogram measure 12 centimeters and 22 centimeters and intersect at an angle of 143 degrees. Find the length of the longer sides of the parallelogram to the nearest tenth of a centimeter.

Solution Let the diagonals of parallelogram $PQRS$ intersect at T . The diagonals of a parallelogram bisect each other. If $PR = 12$, then $PT = 6$ and if $QS = 22$, then $QT = 11$. Let PQ be the longer side of the parallelogram, the side opposite the larger angle at which the diagonals intersect. Therefore, $m\angle PTQ = 143$. Write the Law of Cosines for $PQ^2 = t^2$ in terms of $QT = p = 11$, $PT = q = 6$, and $\cos T = \cos 143^\circ$.



$$\begin{aligned} t^2 &= p^2 + q^2 - 2pq \cos T \\ &= 11^2 + 6^2 - 2(11)(6) \cos 143^\circ \\ &= 121 + 36 - 132 \cos 143^\circ \\ t &= \sqrt{157 - 132 \cos 143^\circ} \end{aligned}$$

Note that $\cos 143^\circ$ is negative so $-132 \cos 143^\circ$ is positive. Using a calculator, we find that $t \approx 16.19937923$.

Answer To the nearest tenth, $PQ = 16.2$ cm.

Exercises**Writing About Mathematics**

1. Explain how the Law of Cosines can be used to show that in an obtuse triangle, the side opposite an obtuse angle is the longest side of the triangle.
2. Explain the relationship between the Law of Cosines and the Pythagorean Theorem.

Developing Skills

3. In $\triangle MAR$, express m^2 in terms of a , r , and $\cos M$.
4. In $\triangle NOP$, express p^2 in terms of n , o , and $\cos P$.
5. In $\triangle ABC$, if $a = 3$, $b = 5$, and $\cos C = \frac{1}{5}$, find the exact value of c .
6. In $\triangle DEF$, if $e = 8$, $f = 3$, and $\cos D = \frac{3}{4}$, find the exact value of d .
7. In $\triangle HIJ$, if $h = 10$, $j = 7$, and $\cos I = 0.6$, find the exact value of i .

In 8–13, find the exact value of the third side of each triangle.

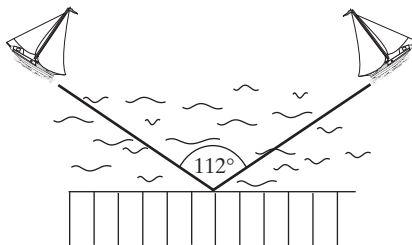
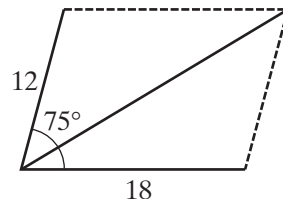
8. In $\triangle ABC$, $b = 4$, $c = 4$, and $m\angle A = \frac{\pi}{3}$.
9. In $\triangle PQR$, $p = 6$, $q = \sqrt{2}$, and $m\angle R = \frac{\pi}{4}$.
10. In $\triangle DEF$, $d = \sqrt{3}$, $e = 5$, and $m\angle F = \frac{\pi}{6}$.
11. In $\triangle ABC$, $a = 6$, $b = 4$, and $m\angle C = \frac{2\pi}{3}$.
12. In $\triangle RST$, $RS = 9$, $ST = 9\sqrt{3}$, and $m\angle S = \frac{5\pi}{6}$.
13. In $\triangle ABC$, $AB = 2\sqrt{2}$, $BC = 4$, and $m\angle B = \frac{3\pi}{4}$.

In 14–19, find, to the nearest tenth, the measure of the third side of each triangle.

14. In $\triangle ABC$, $b = 12.4$, $c = 8.70$, and $m\angle A = 23$.
15. In $\triangle PQR$, $p = 126$, $q = 214$, and $m\angle R = 42$.
16. In $\triangle DEF$, $d = 3.25$, $e = 5.62$, and $m\angle F = 58$.
17. In $\triangle ABC$, $a = 62.5$, $b = 44.7$, and $m\angle C = 133$.
18. In $\triangle RST$, $RS = 0.375$, $ST = 1.29$, and $m\angle S = 167$.
19. In $\triangle ABC$, $AB = 2.35$, $BC = 6.24$, and $m\angle B = 115$.

Applying Skills

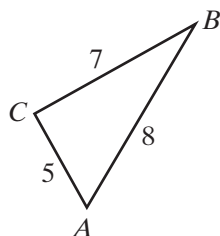
20. Ann and Bill Bekebrede follow a familiar triangular path when they take a walk. They walk from home for 0.52 mile along a straight road, turn at an angle of 95° , walk for another 0.46 mile, and then return home.
- Find, to the nearest hundredth of a mile, the length of the last portion of their walk.
 - Find, to the nearest hundredth of a mile, the total distance that they walk.
21. When two forces act on an object, the resultant force is the single force that would have produced the same result. When the magnitudes of the two forces are represented by the lengths of two sides of a parallelogram, the resultant can be represented by the length of the diagonal of the parallelogram. If forces of 12 pounds and 18 pounds act at an angle of 75° , what is the magnitude of the resultant force to the nearest hundredth pound?
22. A field is in the shape of a parallelogram. The lengths of two adjacent sides are 48 meters and 65 meters. The measure of one angle of the parallelogram is 100° .
- Find, to the nearest meter, the length of the longer diagonal.
 - Find, to the nearest meter, the length of the shorter diagonal.
23. A pole is braced by two wires that extend from the top of the pole to the ground. The lengths of the wires are 16 feet and 18 feet and the measure of the angle between the wires is 110° . Find, to the nearest foot, the distance between the points at which the wires are fastened to the ground.
24. Two points A and B are on the shoreline of Lake George. A surveyor is located at a third point C some distance from both points. The distance from A to C is 180.0 meters and the distance from B to C is 120.0 meters. The surveyor determines that the measure of $\angle ACB$ is 56.3° . To the nearest tenth of a meter, what is the distance from A to B ?
25. Two sailboats leave a dock at the same time sailing on courses that form an angle of 112° with each other. If one boat sails at 10.0 knots per hour and the other sails at 12.0 knots per hour, how many nautical miles apart are the boats after two hours? (nautical miles = knots \times time) Round to the nearest tenth.



26. Use the Law of Cosines to prove that if the angle between two congruent sides of a triangle measures 60° , the triangle is equilateral.

14-3 USING THE LAW OF COSINES TO FIND ANGLE MEASURE

The measures of three sides of a triangle determine the size and shape of the triangle. If we know the measures of three sides of a triangle, we can use the Law of Cosines to find the measure of any angle of the triangle. For example, in $\triangle ABC$, if $a = 7$, $b = 5$, and $c = 8$, use the Law of Cosines to find $\cos A$.



$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$7^2 = 5^2 + 8^2 - 2(5)(8) \cos A$$

$$49 = 25 + 64 - 80 \cos A$$

$$80 \cos A = 25 + 64 - 49$$

$$80 \cos A = 40$$

$$\cos A = \frac{40}{80}$$

$$\cos A = \frac{1}{2}$$

Since A is an angle of a triangle, $0^\circ < A < 180^\circ$. Therefore, $A = 60^\circ$.

The steps used to solve for $\cos A$ in terms of the measures of the sides can be applied to the general formula of the Law of Cosines to express the cosine of any angle of the triangle in terms of the lengths of the sides.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

This formula can be rewritten in terms of the cosine of any angle of $\triangle ABC$.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

EXAMPLE 1

In $\triangle ABC$, $a = 12$, $b = 8$, $c = 6$. Find $\cos C$.

Solution

How to Proceed

- (1) Write the Law of Cosines in terms of $\cos C$:
- (2) Substitute the given values:
- (3) Perform the computation. Reduce the fractional value of $\cos C$ to lowest terms:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos C = \frac{12^2 + 8^2 - 6^2}{2(12)(8)}$$

$$\cos C = \frac{144 + 64 - 36}{192}$$

$$= \frac{172}{192}$$

$$= \frac{43}{48}$$

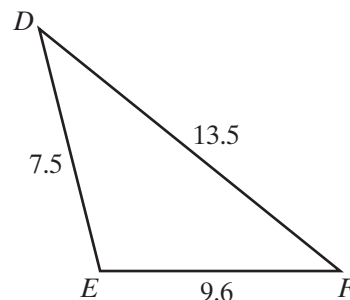
Answer $\cos C = \frac{43}{48}$

EXAMPLE 2

Find, to the nearest degree, the measure of the largest angle of $\triangle DEF$ if $DE = 7.5$, $EF = 9.6$, and $DF = 13.5$.

Solution The largest angle of the triangle is opposite the longest side. The largest angle is $\angle E$, the angle opposite the longest side, DF . Let $DE = f = 7.5$, $EF = d = 9.6$, and $DF = e = 13.5$. Write the formula in terms of $\cos E$.

$$\begin{aligned}\cos E &= \frac{d^2 + f^2 - e^2}{2df} \\ &= \frac{9.6^2 + 7.5^2 - 13.5^2}{2(9.6)(7.5)} \\ &\approx -0.235\end{aligned}$$



Therefore, $E = \arccos(-0.235)$. Use a calculator to find the arccosine:

ENTER: **2nd** **COS⁻¹** **(-)** 0.235
) **ENTER**

DISPLAY: $\cos^{-1}(-0.235)$
103.5916228

Answer $m\angle E = 104$

Exercises

Writing About Mathematics

1. Explain how the Law of Cosines can be used to show that 4, 7, and 12 cannot be the measures of the sides of a triangle.
2. Show that if $\angle C$ is an obtuse angle, $a^2 + b^2 < c^2$.

Developing Skills

3. In $\triangle TUV$, express $\cos T$ in terms of t , u , and v .
4. In $\triangle PQR$, express $\cos Q$ in terms of p , q , and r .
5. In $\triangle KLM$, if $k = 4$, $l = 5$, and $m = 8$, find the exact value of $\cos M$.
6. In $\triangle XYZ$, if $x = 1$, $y = 2$, and $z = \sqrt{5}$, find the exact value of $\cos Z$.

In 7–12, find the cosine of each angle of the given triangle.

7. In $\triangle ABC$, $a = 4$, $b = 6$, $c = 8$.
8. In $\triangle ABC$, $a = 12$, $b = 8$, $c = 8$.
9. In $\triangle DEF$, $d = 15$, $e = 12$, $f = 8$.
10. In $\triangle PQR$, $p = 2$, $q = 4$, $r = 5$.
11. In $\triangle MNP$, $m = 16$, $n = 15$, $p = 8$.
12. In $\triangle ABC$, $a = 5$, $b = 12$, $c = 13$.

In 13–18, find, to the nearest degree, the measure of each angle of the triangle with the given measures of the sides.

- 13.** 12, 20, 22 **14.** 9, 10, 15 **15.** 30, 35, 45 **16.** 11, 11, 15 **17.** 32, 40, 38 **18.** 7, 24, 25

Applying Skills

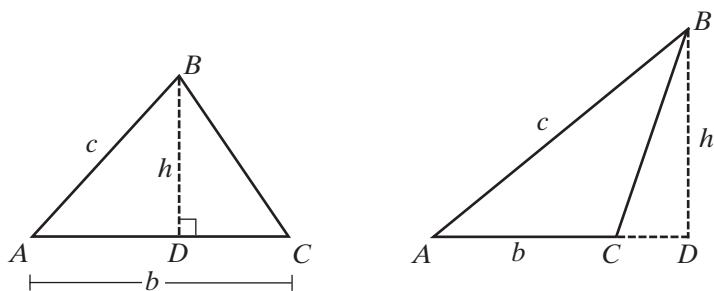
- 19.** Two lighthouses are 12 miles apart along a straight shore. A ship is 15 miles from one lighthouse and 20 miles from the other. Find, to the nearest degree, the measure of the angle between the lines of sight from the ship to each lighthouse.
- 20.** A tree is braced by wires 4.2 feet and 4.7 feet long that are fastened to the tree at the same point and to the ground at points 7.8 feet apart. Find, to the nearest degree, the measure of the angle between the wires at the tree.
- 21.** A kite is in the shape of a quadrilateral with two pair of congruent adjacent sides. The lengths of two sides are 20.0 inches and the lengths of the other two sides are 35.0 inches. The two shorter sides meet at an angle of 115° .
- Find the length of the diagonal between the points at which the unequal sides meet. Write the length to the nearest tenth of an inch.
 - Using the answer to part **a**, find, to the nearest degree, the measure of the angle at which the two longer sides meet.
- 22.** A beam 16.5 feet long supports a roof with rafters each measuring 12.4 feet long. What is the measure of the angle at which the rafters meet?



- 23.** A walking trail is laid out in the shape of a triangle. The lengths of the three paths that make up the trail are 2,500 meters, 2,000 meters, and 1,800 meters. Determine, to the nearest degree, the measure of the greatest angle of the trail.
- 24.** Use the formula $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ to show that the measure of each angle of an equilateral triangle is 60° .

14-4 AREA OF A TRIANGLE

When the measures of two sides and the included angle of a triangle are known, the size and shape of the triangle is determined. Therefore, it is possible to use these known values to find the area of the triangle. Let $\triangle ABC$ be any triangle. If we know the measures of $AB = c$, $AC = b$, and the included angle, $\angle A$, we can find the area of the triangle.

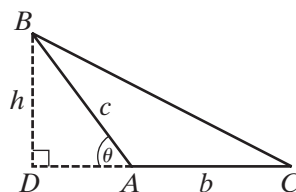


In $\triangle ABC$ let $\angle A$ be an acute angle, \overline{BD} be the altitude from B to \overleftrightarrow{AC} , and $BD = h$. In right $\triangle ABD$, $\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BD}{AB} = \frac{h}{c}$ or $h = c \sin A$. Therefore:

$$\text{Area of } \triangle ABC = \frac{1}{2}bh = \frac{1}{2}bc \sin A$$

What if $\angle A$ is an obtuse angle?

Let $\angle A$ be an obtuse angle of $\triangle ABC$, \overline{BD} be the altitude from B to \overleftrightarrow{AC} and $BD = h$. In right $\triangle ABD$, let $m\angle DAB = \theta$. Then $\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{BD}{AB} = \frac{h}{c}$ or $h = c \sin \theta$. Therefore:



$$\text{Area of } \triangle ABC = \frac{1}{2}bh = \frac{1}{2}bc \sin \theta$$

Since $\angle DAB$ and $\angle BAC$ are adjacent angles whose sum is a straight angle, $m\angle DAB = 180 - m\angle BAC$ and $\sin \theta = \sin A$. Therefore, the area of $\triangle ABC$ is again equal to $\frac{1}{2}bc \sin A$. Thus, for any angle, we have shown that:

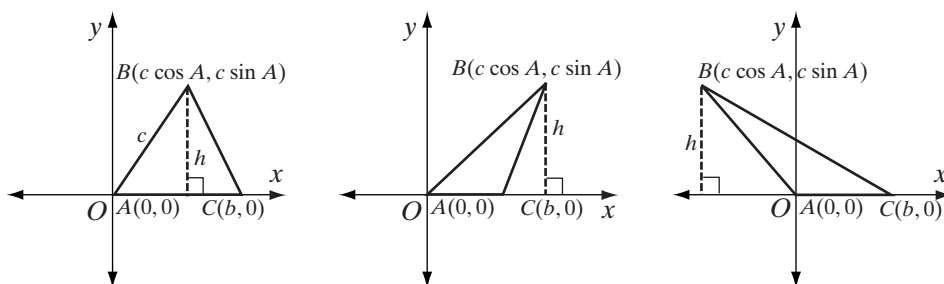
$$\text{Area of } \triangle ABC = \frac{1}{2}bh = \frac{1}{2}bc \sin A$$

The area of a triangle is equal to one-half the product of the measures of two sides of the triangle times the sine of the measure of the included angle. This formula can be written in terms of any two sides and the included angle.

$$\text{Area } \triangle ABC = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

Triangles in the Coordinate Plane

When a triangle is drawn in the coordinate plane, the area formula follows easily. Let $\triangle ABC$ be any triangle. Place the triangle with $A(0, 0)$ at the origin and $C(b, 0)$ on the positive ray of the x -axis, and $\angle A$ an angle in standard position.



From Section 14-1, we know that the coordinates of B are $(c \cos A, c \sin A)$. For each triangle, h is the length of the perpendicular from B to the x -axis and $h = c \sin A$. Therefore,

$$\text{Area } \triangle ABC = \frac{1}{2}bh = \frac{1}{2}bc \sin A$$

EXAMPLE 1

Find the area of $\triangle DEF$ if $DE = 14$, $EF = 9$, and $m\angle E = 30^\circ$.

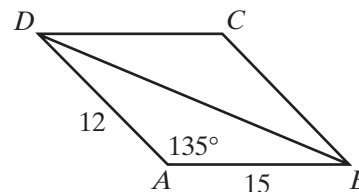
Solution $DE = f = 14$, $EF = d = 9$, and $m\angle E = 30^\circ$.

$$\text{Area of } \triangle DEF = \frac{1}{2}df \sin E = \frac{1}{2}(9)(14)\left(\frac{1}{2}\right) = \frac{63}{2} = 31\frac{1}{2} \quad \text{Answer}$$

EXAMPLE 2

The adjacent sides of parallelogram $ABCD$ measure 12 and 15. The measure of one angle of the parallelogram is 135° . Find the area of the parallelogram.

Solution The diagonal of a parallelogram separates the parallelogram into two congruent triangles. Draw diagonal \overline{BD} . In $\triangle DAB$, $DA = b = 12$, $AB = d = 15$, and $m\angle A = 135^\circ$.



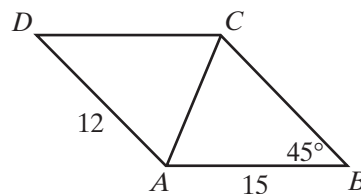
$$\text{Area of } \triangle DAB = \frac{1}{2}bd \sin A = \frac{1}{2}(12)(15) \sin 135^\circ = \frac{1}{2}(12)(15)\left(\frac{\sqrt{2}}{2}\right) = 45\sqrt{2}$$

$$\text{Area of } \triangle DBC = \text{Area of } \triangle DAB = 45\sqrt{2}$$

$$\text{Area of parallelogram } ABCD = \text{Area of } \triangle DBC + \text{Area of } \triangle DAB = 90\sqrt{2}$$

Answer $90\sqrt{2}$ square units

Note: The same answer to Example 2 is obtained if we use adjacent angle B or D . Consecutive angles of a parallelogram are supplementary. If $m\angle A = 135$, then $m\angle B = 180 - 135 = 45$. Opposite sides of a parallelogram are congruent. If $DA = 12$, then $BC = 12$. Draw diagonal \overline{AC} . In $\triangle ABC$, $BC = a = 12$, $AB = c = 15$, and $m\angle B = 45$.



$$\text{Area of } \triangle ABC = \frac{1}{2}ac \sin B = \frac{1}{2}(12)(15) \sin 45^\circ = \frac{1}{2}(12)(15)\left(\frac{\sqrt{2}}{2}\right) = 45\sqrt{2}$$

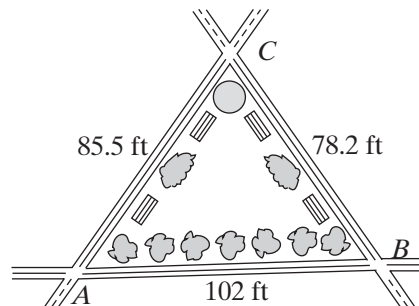
$$\text{Area of } \triangle CDA = \text{Area of } \triangle ABC = 45\sqrt{2}$$

$$\text{Area of parallelogram } ABCD = \text{Area of } \triangle ABC + \text{Area of } \triangle CDA = 90\sqrt{2}$$

EXAMPLE 3

Three streets intersect in pairs enclosing a small triangular park. The measures of the distances between the intersections are 85.5 feet, 102 feet, and 78.2 feet. Find the area of the park to the nearest ten square feet.

Solution Let A , B , and C be the intersections of the streets, forming $\triangle ABC$. Use the Law of Cosines to find the measure of any angle, for example, $\angle A$. Then use the formula for the area of a triangle in terms of the measures of two sides and an angle.



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{85.5^2 + 102^2 - 78.2^2}{2(85.5)(102)} \approx 0.6650$$

Use a calculator to find the measure of $\angle A$.

$$m\angle A = \cos^{-1} 0.6650 \approx 48.31$$

$$\text{Area of } \triangle ABC = \frac{1}{2}bc \sin A = \frac{1}{2}(85.5)(102)(\sin 48.31^\circ) \approx 3,256$$

Answer The area of the park is approximately 3,260 square feet. ■

Exercises**Writing About Mathematics**

1. Rosa found the area of parallelogram $ABCD$ by using $(AB)(BC)(\sin B)$. Riley found the area of parallelogram $ABCD$ by using $(AB)(BC)(\sin A)$. Explain why Rosa and Riley both got the correct answer.
2. Jessica said that the area of rhombus $PQRS$ is $(PQ)^2(\sin P)$. Do you agree with Jessica? Explain why or why not.

Developing Skills

In 3–8, find the area of each $\triangle ABC$.

- | | |
|--|---|
| 3. $b = 3, c = 8, \sin A = \frac{1}{4}$ | 4. $a = 12, c = 15, \sin B = \frac{1}{3}$ |
| 5. $b = 9, c = 16, \sin A = \frac{5}{6}$ | 6. $a = 24, b = 12, \sin C = \frac{3}{4}$ |
| 7. $b = 7, c = 8, \sin A = \frac{3}{5}$ | 8. $a = 10, c = 8, \sin B = \frac{3}{10}$ |

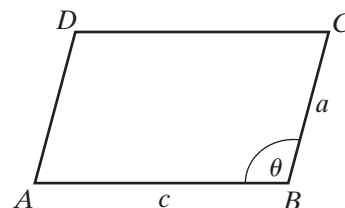
In 9–14, find the area of each triangle to the nearest tenth.

9. In $\triangle ABC$, $b = 14.6, c = 12.8, m\angle A = 56$.
10. In $\triangle ABC$, $a = 326, c = 157, m\angle B = 72$.
11. In $\triangle DEF$, $d = 5.83, e = 5.83, m\angle F = 48$.
12. In $\triangle PQR$, $p = 212, q = 287, m\angle R = 124$.
13. In $\triangle RST$, $t = 15.7, s = 15.7, m\angle R = 98$.
14. In $\triangle DEF$, $e = 336, f = 257, m\angle D = 122$.
15. Find the exact value of the area of an equilateral triangle if the length of one side is 40 meters.
16. Find the exact value of the area of an isosceles triangle if the measure of a leg is 12 centimeters and the measure of the vertex angle is 45 degrees.
17. Find the area of a parallelogram if the measures of two adjacent sides are 40 feet and 24 feet and the measure of one angle of the parallelogram is 30 degrees.

Applying Skills

18. A field is bordered by two pairs of parallel roads so that the shape of the field is a parallelogram. The lengths of two adjacent sides of the field are 2 kilometers and 3 kilometers, and the length of the shorter diagonal of the field is 3 kilometers.
 - a. Find the cosine of the acute angle of the parallelogram.
 - b. Find the exact value of the sine of the acute angle of the parallelogram.
 - c. Find the exact value of the area of the field.
 - d. Find the area of the field to the nearest integer.

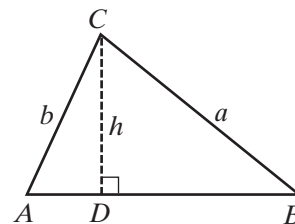
19. The roof of a shed consists of four congruent isosceles triangles. The length of each equal side of one triangular section is 22.0 feet and the measure of the vertex angle of each triangle is 75° . Find, to the nearest square foot, the area of one triangular section of the roof.
20. A garden is in the shape of an isosceles trapezoid. The lengths of the parallel sides of the garden are 30 feet and 20 feet, and the length of each of the other two sides is 10 feet. If a base angle of the trapezoid measures 60° , find the exact area of the garden.
21. In $\triangle ABC$, $m\angle B = 30$ and in $\triangle DEF$, $m\angle E = 150$. Show that if $AB = DE$ and $BC = EF$, the areas of the two triangles are equal.
22. Aaron wants to draw $\triangle ABC$ with $AB = 15$ inches, $BC = 8$ inches, and an area of 40 square inches.
- What must be the sine of $\angle B$?
 - Find, to the nearest tenth of a degree, the measure of $\angle B$.
 - Is it possible for Aaron to draw two triangles that are not congruent to each other that satisfy the given conditions? Explain.
23. Let $ABCD$ be a parallelogram with $AB = c$, $BC = a$, and $m\angle B = \theta$.
- Write a formula for the area of parallelogram $ABCD$ in terms of c , a , and θ .
 - For what value of θ does parallelogram $ABCD$ have the greatest area?



14-5 LAW OF SINES

If we know the measures of two angles and the included side of a triangle (ASA), or if we know the measures of two angles and the side opposite one of the angles of a triangle (AAS), the size and shape of the triangle is determined. Therefore, we should be able to find the measures of the remaining sides.

In $\triangle ABC$, let $m\angle A$ and $m\angle B$ be two angles and $AC = b$ be the side opposite one of the angles. When we know these measures, is it possible to find $BC = a$?



Let \overline{CD} be the altitude from C to \overline{AB} . Let $CD = h$ and $BC = a$.

In right $\triangle ACD$,

$$\sin A = \frac{\text{opp}}{\text{hyp}}$$

$$\sin A = \frac{h}{b}$$

$$h = b \sin A$$

In right $\triangle BCD$,

$$\sin B = \frac{\text{opp}}{\text{hyp}}$$

$$\sin B = \frac{h}{a}$$

$$h = a \sin B$$

Since $a \sin B$ and $b \sin A$ are each equal to h , they are equal to each other. Therefore, $a \sin B = b \sin A$. To solve for a , divide both sides of this equation by $\sin B$.

$$a \sin B = b \sin A$$

$$\frac{a \sin B}{\sin B} = \frac{b \sin A}{\sin B}$$

$$a = \frac{b \sin A}{\sin B}$$

More generally, we can establish a proportional relationship between two angles and the sides opposite these angles in a triangle. Divide both sides of this equation by $\sin A \sin B$.

$$a \sin B = b \sin A$$

$$\frac{a \sin B}{\sin A \sin B} = \frac{b \sin A}{\sin A \sin B}$$

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

An alternative derivation of this formula begins with the formulas for the area of a triangle.

$$\text{Area } \triangle ABC = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

We can multiply each of the last three terms of this equality by 2.

$$2\left(\frac{1}{2}bc \sin A\right) = 2\left(\frac{1}{2}ac \sin B\right) = 2\left(\frac{1}{2}ab \sin C\right)$$

$$bc \sin A = ac \sin B = ab \sin C$$

Now divide each side of the equality by abc .

$$\frac{bc \sin A}{abc} = \frac{ac \sin B}{abc} = \frac{ab \sin C}{abc}$$

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

These equal ratios are usually written in terms of their reciprocals.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

This equality is called the **Law of Sines**.

EXAMPLE 1

In $\triangle ABC$, $c = 12$, $m\angle B = 120$, and $m\angle C = 45$. Find the exact value of b .

Solution *How to Proceed*

- (1) Use the ratios of the Law of Sines that use b and c :

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

- (2) Substitute the given values:

$$\frac{b}{\sin 120^\circ} = \frac{12}{\sin 45^\circ}$$

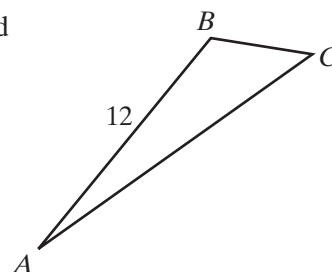
- (3) Solve for b , substituting sine values:

$$b = \frac{12 \sin 120^\circ}{\sin 45^\circ}$$

$$b = \frac{12\left(\frac{\sqrt{3}}{2}\right)}{\frac{\sqrt{2}}{2}}$$

- (4) Write the value of b in simplest form:

$$\begin{aligned} b &= \frac{12\sqrt{3}}{2} \times \frac{2}{\sqrt{2}} \\ &= \frac{12\sqrt{3}}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{12\sqrt{6}}{2} \\ &= 6\sqrt{6} \end{aligned}$$



Answer $b = 6\sqrt{6}$

EXAMPLE 2

In $\triangle DEF$, $m\angle D = 50$, $m\angle E = 95$, and $f = 12.6$. Find d to the nearest tenth.

Solution Use the form of the Law of Sines in terms of the side whose measure is known, f , and the side whose measure is to be found, d .

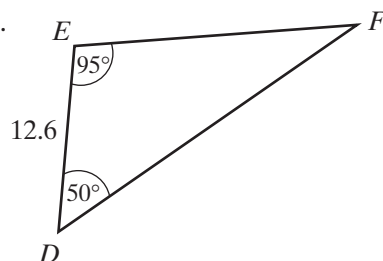
$$\frac{d}{\sin D} = \frac{f}{\sin F}$$

To use this formula, we need to know $m\angle F$.

$$m\angle F = 180 - (50 + 95) = 35$$

Therefore:

$$\begin{aligned} \frac{d}{\sin 50^\circ} &= \frac{12.6}{\sin 35^\circ} \\ d &= \frac{12.6 \sin 50^\circ}{\sin 35^\circ} \end{aligned}$$



Use a calculator to evaluate d .

ENTER: (12.6 SIN 50))
 ÷ SIN 35) ENTER

DISPLAY: {12.6 sin(50)}/sin(35)
 16.82802739

Answer To the nearest tenth, $d = 16.8$.

Exercises

Writing About Mathematics

1. If the sine of an angle of a triangle is known, is it possible to determine the measure of the angle? Explain why or why not.
2. If the cosine of an angle of a triangle is known, is it possible to determine the measure of the angle? Explain why or why not.

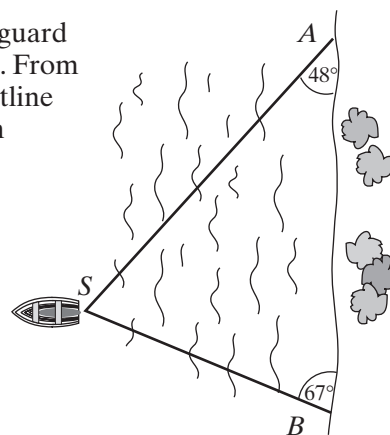
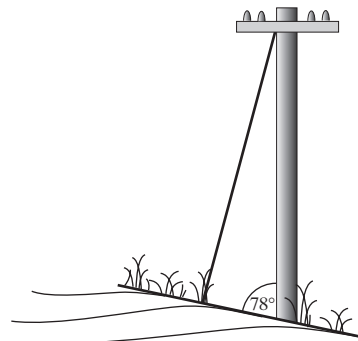
Developing Skills

3. In $\triangle ABC$, if $a = 9$, $m\angle A = \frac{\pi}{3}$, and $m\angle B = \frac{\pi}{4}$, find the exact value of b in simplest form.
4. In $\triangle ABC$, if $a = 24$, $m\angle A = \frac{\pi}{6}$, and $m\angle B = \frac{\pi}{2}$, find the exact value of b in simplest form.
5. In $\triangle ABC$, if $c = 12$, $m\angle C = \frac{2\pi}{3}$, and $m\angle B = \frac{\pi}{6}$, find the exact value of b in simplest form.
6. In $\triangle ABC$, if $b = 8$, $m\angle A = \frac{\pi}{3}$, and $m\angle C = \frac{5\pi}{12}$, find the exact value of a in simplest form.
7. In $\triangle DEF$, $\sin D = 0.4$, $\sin E = 0.25$, and $d = 20$. Find the exact value of e .
8. In $\triangle PQR$, $\sin P = \frac{3}{4}$, $\sin R = \frac{2}{5}$, and $p = 40$. Find the exact value of r .
9. In $\triangle DEF$, $m\angle D = 47$, $m\angle E = 84$, and $d = 17.3$. Find e to the nearest tenth.
10. In $\triangle DEF$, $m\angle D = 56$, $m\angle E = 44$, and $d = 37.5$. Find e to the nearest tenth.
11. In $\triangle LMN$, $m\angle M = 112$, $m\angle N = 54$, and $m = 51.0$. Find n to the nearest tenth.
12. In $\triangle ABC$, $m\angle A = 102$, $m\angle B = 34$, and $a = 25.8$. Find c to the nearest tenth.
13. In $\triangle PQR$, $m\angle P = 125$, $m\angle Q = 14$, and $p = 122$. Find r to the nearest integer.
14. In $\triangle RST$, $m\angle R = 12$, $m\angle S = 75$, and $r = 3.52$. Find t to the nearest tenth.
15. In $\triangle CDE$, $m\angle D = 125$, $m\angle E = 28$, and $d = 12.5$. Find c to the nearest hundredth.
16. The base of an isosceles triangle measures 14.5 centimeters and the vertex angle measures 110 degrees.
 - a. Find the measure of one of the congruent sides of the triangle to the nearest hundredth.
 - b. Find the perimeter of the triangle to the nearest tenth.

17. The length of one of the equal sides of an isosceles triangle measures 25.8 inches and each base angle measures 53 degrees.
- Find the measure of the base of the triangle to the nearest tenth.
 - Find the perimeter of the triangle to the nearest inch.
18. Use the Law of Sines to show that if $\angle C$ of $\triangle ABC$ is a right angle, $\sin A = \frac{a}{c}$.

Applying Skills

19. A telephone pole on a hillside makes an angle of 78 degrees with the upward slope. A wire from the top of the pole to a point up the hill is 12.0 feet long and makes an angle of 15 degrees with the pole.
- Find, to the nearest hundredth, the distance from the foot of the pole to the point at which the wire is fastened to the ground.
 - Use the answer to part **a** to find, to the nearest tenth, the height of the pole.
20. Three streets intersect in pairs enclosing a small park. Two of the angles at which the streets intersect measure 85 degrees and 65 degrees. The length of the longest side of the park is 275 feet. Find the lengths of the other two sides of the park.
21. On the playground, the 10-foot ladder to the top of the slide makes an angle of 48 degrees with the ground. The slide makes an angle of 32 degrees with the ground.
- How long is the slide to the nearest tenth?
 - What is the distance from the foot of the ladder to the foot of the slide to the nearest tenth?
22. A distress signal from a ship, S , is received by two coast guard stations located 3.8 miles apart along a straight coastline. From station A , the signal makes an angle of 48° with the coastline and from station B the signal makes an angle of 67° with the coastline. Find, to the nearest tenth of a mile, the distance from the ship to the nearer station.



23. Two sides of a triangular lot form angles that measure 29.1° and 33.7° with the third side, which is 487 feet long. To the nearest dollar, how much will it cost to fence the lot if the fencing costs \$5.59 per foot?

14-6 THE AMBIGUOUS CASE

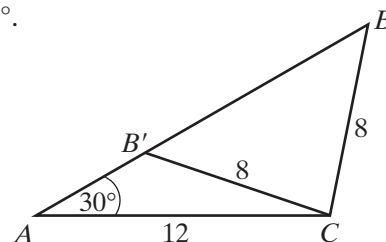
If we know the measures of two sides of a triangle and the angle opposite one of them, the Law of Sines makes it possible for us to find the sine of the angle opposite the second side whose measure is known. However, we know that the measures of two sides and the angle opposite one of them (SSA) is not sufficient to determine the size and shape of the triangle in every case. This is often called the **ambiguous case**.

Consider the following cases in which we are given a , b , and angle A . For $0 < \sin B < 1$, there are two values of B in the interval from 0° to 180° . We will call these values B and B' . Since the sum of the degree measures of the angles of a triangle is 180, the sum of the degree measures of two angles of a triangle must be less than 180.

CASE 1 *Two triangles can be drawn.*

In $\triangle ABC$, $a = 8$, $b = 12$, and $\angle A = 30^\circ$.
We can use the Law of Sines to find $\sin B$.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{8}{\sin 30^\circ} &= \frac{12}{\sin B} \\ 8 \sin B &= 12 \sin 30^\circ \\ \sin B &= \frac{12(\frac{1}{2})}{8} \\ \sin B &= \frac{3}{4}\end{aligned}$$

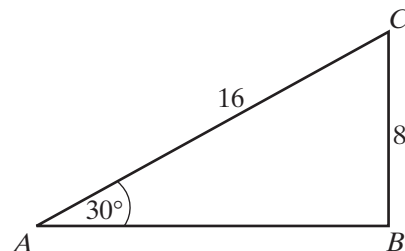


When $\sin B = \frac{3}{4}$, $m\angle B \approx 48.59$ or $m\angle B' \approx 180 - 48.59 = 131.41$. As shown in the diagram, there are two triangles, $\triangle ABC$ and $\triangle AB'C$ in which two sides measure 8 and 12 and the angle opposite the shorter of these sides measures 30° . **Two triangles can be drawn.**

CASE 2 *Only one triangle can be drawn and that triangle is a right triangle.*

In $\triangle ABC$, $a = 8$, $b = 16$, and $\angle A = 30^\circ$.
We can use the Law of Sines to find $\sin B$.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{8}{\sin 30^\circ} &= \frac{16}{\sin B} \\ 8 \sin B &= 16 \sin 30^\circ \\ \sin B &= \frac{16(\frac{1}{2})}{8} \\ \sin B &= 1\end{aligned}$$



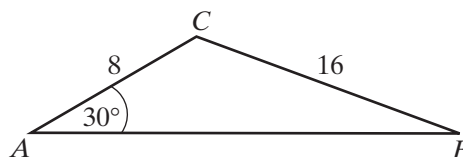
When $\sin B = 1$, $m\angle B = 90$. This is the only measure of $\angle B$ that can be the measure of an angle of a triangle. **One triangle can be drawn and that triangle is a right triangle.**

Note: If $m\angle A = 150$, $\sin B = 1$ and $m\angle B = 90$. There is no triangle with an obtuse angle and a right angle.

CASE 3 *Only one triangle can be drawn.*

In $\triangle ABC$, $a = 16$, $b = 8$, and $\angle A = 30^\circ$. We can use the Law of Sines to find $\sin B$.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{16}{\sin 30^\circ} &= \frac{8}{\sin B} \\ 16 \sin B &= 8 \sin 30^\circ \\ \sin B &= \frac{8(\frac{1}{2})}{16} \\ \sin B &= \frac{1}{4}\end{aligned}$$



When $\sin B = \frac{1}{4}$, $m\angle B = 14.48$ or $m\angle B = 180 - 14.48 = 165.52$.

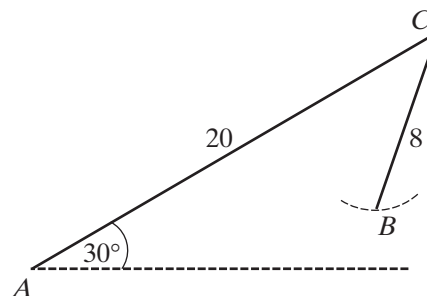
- If we let $\angle B$ be an acute angle, $m\angle A + m\angle B = 30 + 14.48 < 180$. There is a triangle with $m\angle A = 30$ and $m\angle B = 14.48$. ✓
- If we let $\angle B$ be an obtuse angle, $m\angle A + m\angle B = 30 + 165.52 > 180$. There is no triangle with $m\angle A = 30$ and $m\angle B = 165.52$. ✗

Only one triangle can be drawn.

CASE 4 *No triangle can be drawn.*

In $\triangle ABC$, $a = 8$, $b = 20$, and $\angle A = 30^\circ$. We can use the Law of Sines to find $\sin B$.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B} \\ \frac{8}{\sin 30^\circ} &= \frac{20}{\sin B} \\ 8 \sin B &= 20 \sin 30^\circ \\ \sin B &= \frac{20(\frac{1}{2})}{8} \\ \sin B &= \frac{5}{4}\end{aligned}$$



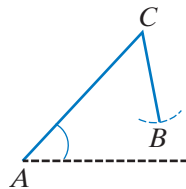
There is no value of B for which $\sin B > 1$. **No triangle can be drawn.**

These four examples show that for the given lengths of two sides and the measure of an acute angle opposite one of them, two, one, or no triangles can be formed.

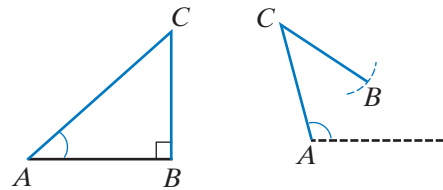
To determine the number of solutions given a , b , and $m\angle A$ in $\triangle ABC$:

Use the Law of Sines to solve for $\sin B$.

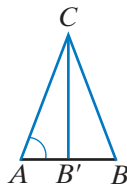
If $\sin B > 1$, there is no triangle.



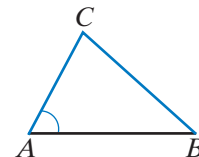
If $\sin B = 1$, there is one right triangle if $\angle A$ is acute but no triangle if $\angle A$ is obtuse.



If $\angle A$ is acute and $\sin B < 1$, find two possible values of $\angle B$:
 $0 < m\angle B < 90$ and $m\angle B' = 180 - m\angle B$.

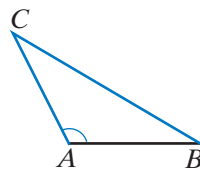


If $m\angle A + m\angle B' < 180$, there are two possible triangles, $\triangle ABC$ and $\triangle AB'C$.

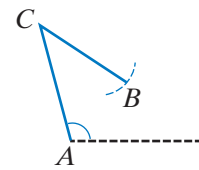


If $m\angle A + m\angle B' \geq 180$, $\triangle AB'C$ is not a triangle. There is only one possible triangle, $\triangle ABC$.

If $\angle A$ is obtuse and $\sin B < 1$, $\angle B$ must be acute: $0 < m\angle B < 90$.

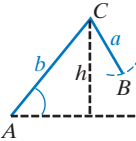
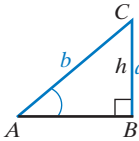
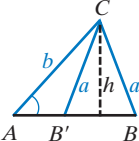
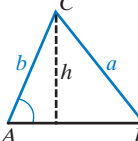
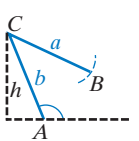
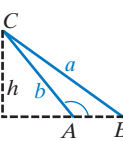


If $m\angle A + m\angle B < 180$, there is one triangle, $\triangle ABC$.



If $m\angle A + m\angle B \geq 180$, there is no triangle.

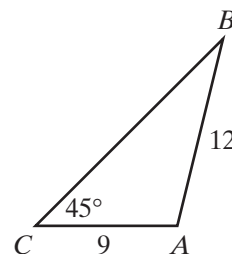
Alternatively, if we let $h = b \sin A$, the height of the triangle, we can summarize the number of possible triangles given a , b , and $m\angle A$ in $\triangle ABC$:

					
$\angle A$ is:	Acute	Acute	Acute	Acute	Obtuse
	$a < h$	$h = a$	$h < a < b$	$a > b$	$a < h$
Possible triangles:	None	One, right \triangle	Two	One	None
					One

EXAMPLE I

In $\triangle ABC$, $b = 9$, $c = 12$, and $m\angle C = 45^\circ$.

- Find the exact value of $\sin B$.
- For the value of $\sin B$ in **a**, find, to the nearest hundredth, the measures of two angles, $\angle B$ and $\angle B'$, that could be angles of a triangle.
- How many triangles are possible?



Solution **a.** (1) Use the ratios of the Law of Sines that use b and c :

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

(2) Substitute the given values:

$$\frac{9}{\sin B} = \frac{12}{\sin 45^\circ}$$

(3) Solve for $\sin B$:

$$12 \sin B = 9 \sin 45^\circ$$

(4) Substitute the exact value of $\sin 45^\circ$:

$$\sin B = \frac{9}{12} \sin 45^\circ$$

$$\sin B = \frac{3}{4} \times \frac{\sqrt{2}}{2}$$

$$\sin B = \frac{3\sqrt{2}}{8}$$

b. Use a calculator to find the approximate measure of $\angle B$.

ENTER: **2nd** **SIN⁻¹** (**3** **×** **2nd** **√** **2** **)** **)** **÷** **8** **ENTER**

DISPLAY: $\sin^{-1}((3 \times \sqrt{2})/8)$
32.02776011

$$m\angle B \approx 32.03 \text{ and } m\angle B' \approx 180 - 32.03 = 147.97 \quad \text{Answer}$$

c. $m\angle A + m\angle B = 45 + 32.03 < 180$ and $\triangle ABC$ is a triangle.

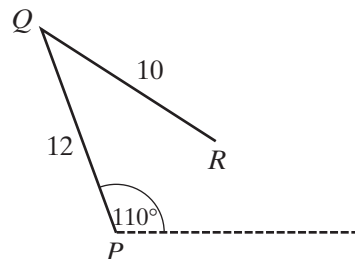
$m\angle A + m\angle B' = 45 + 147.97 > 180$ and $\triangle AB'C$ is *not* a triangle.

There is one possible triangle. **Answer**

EXAMPLE 2

How many triangles can be drawn if the measures of two of the sides are 12 inches and 10 inches and the measure of the angle opposite the 10-inch side is 110 degrees?

Solution Let a possible triangle be $\triangle PQR$ with $PQ = r = 12$ and $QR = p = 10$. The angle opposite \overline{QR} is $\angle P$ and $m\angle P = 110^\circ$.



- (1) We know p , r , and $\angle P$. Use the Law of Sines to solve for $\sin R$:

$$\frac{p}{\sin P} = \frac{r}{\sin R}$$

$$\frac{10}{\sin 110^\circ} = \frac{12}{\sin R}$$

$$10 \sin R = 12 \sin 110^\circ$$

- (2) Use a calculator to find $\sin R$:

$$\sin R = \frac{12 \sin 110^\circ}{10}$$

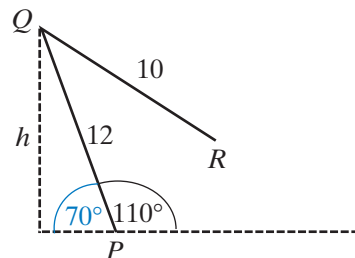
$$\sin R \approx 1.127631145$$

Since $\sin R > 1$, there is no triangle.

Alternative Solution Let h represent the height of the triangle. Then

$$h = r \sin P = 12 \sin 70^\circ \approx 11.28$$

The height of the triangle, or the altitude at Q , would be longer than side \overline{QR} . No such triangle exists.



Answer 0

Exercises

Writing About Mathematics

- Without using formulas that include the sine of an angle, is it possible to determine from the given information in Example 2 that there can be no possible triangle? Justify your answer.
- Explain why, when the measures of two sides and an obtuse angle opposite one of them are given, it is never possible to construct two different triangles.

Developing Skills

In 3–14: **a.** Determine the number of possible triangles for each set of given measures. **b.** Find the measures of the three angles of each possible triangle. Express approximate values to the nearest degree.

- | | |
|--|---|
| 3. $a = 8, b = 10, m\angle A = 20$ | 4. $a = 5, b = 10, m\angle A = 30$ |
| 5. $b = 12, c = 10, m\angle B = 49$ | 6. $a = 6, c = 10, m\angle A = 45$ |
| 7. $c = 18, b = 10, m\angle C = 120$ | 8. $a = 9, c = 10, m\angle C = 150$ |
| 9. $AB = 14, BC = 21, m\angle C = 75$ | 10. $DE = 24, EF = 18, m\angle D = 15$ |
| 11. $PQ = 12, PR = 15, m\angle R = 100$ | 12. $BC = 12, AC = 12\sqrt{2}, m\angle B = 135$ |
| 13. $RS = 3\sqrt{3}, ST = 3, m\angle T = 60$ | 14. $a = 8, b = 10, m\angle A = 45$ |

Applying Skills

15. A ladder that is 15 feet long is placed so that it reaches from level ground to the top of a vertical wall that is 13 feet high.
 - a. Use the Law of Sines to find the angle that the ladder makes with the ground to the nearest hundredth.
 - b. Is more than one position of the ladder possible? Explain your answer.
16. Max has a triangular garden. He measured two sides of the garden and the angle opposite one of these sides. He said that the two sides measured 5 feet and 8 feet and that the angle opposite the 8-foot side measured 75 degrees. Can a garden exist with these measurements? Could there be two gardens of different shapes with these measurements? Write the angle measures and lengths of the sides of the garden(s) if any.
17. Emily wants to draw a parallelogram with the measure of one side 12 centimeters, the measure of one diagonal 10 centimeters and the measure of one angle 120 degrees. Is this possible? Explain why or why not.
18. Ross said that when he jogs, his path forms a triangle. Two sides of the triangle are 2.0 kilometers and 2.5 kilometers in length and the angle opposite the shorter side measures 45 degrees. Rosa said that when she jogs, her path also forms a triangle with two sides of length 2.0 kilometers and 2.5 kilometers and an angle of 45 degrees opposite the shorter side. Rosa said that her route is longer than the route Ross follows. Is this possible? Explain your answer.

14-7 SOLVING TRIANGLES

If the known measures of any three parts of a triangle include at least one side, the measures of the remaining three parts can be determined.

The Right Triangle

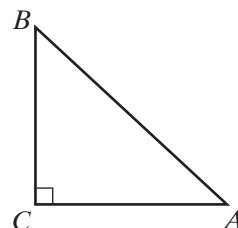
When the triangle is a right triangle, the ratio of the measures of any two sides of the triangle is a trigonometric function value of one of the acute angles.

In right $\triangle ABC$ with $m\angle C = 90^\circ$:

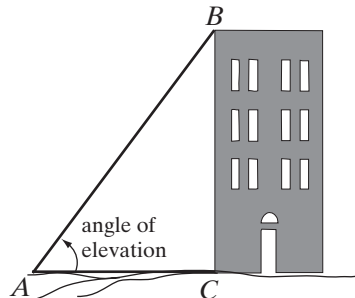
$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC}$$

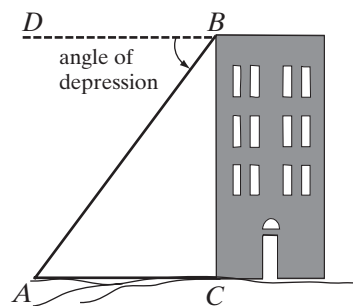


The height of a building, a tree, or any similar object is measured as the length of the perpendicular from the top of the object to the ground. The measurement of height often involves right triangles.



An **angle of elevation** is an angle such that one ray is part of a horizontal line and the other ray represents a line of sight raised upward from the horizontal. To visualize the angle of elevation of a building, think of some point on the same horizontal line as the base of the building. The angle of elevation is the angle through which our line of sight would rotate from the base of the building to its top. In the diagram, $\angle A$ is the angle of elevation.

An **angle of depression** is an angle such that one ray is part of a horizontal line and the other ray represents a line of sight moved downward from the horizontal. To visualize the angle of depression of a building, think of some point on the same horizontal line as the top of the building. The angle of depression is the angle through which our line of sight would rotate from the top of the building to its base. In the diagram, $\angle ABD$ is the angle of depression.



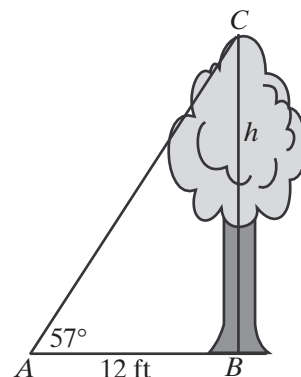
When solving right triangles, we can use the ratio of sides given above or we can use the Law of Sines or the Law of Cosines.

EXAMPLE I

From a point 12 feet from the foot of the tree, the measure of the angle of elevation to the top of the tree is 57° . Find the height of the tree to the nearest tenth of a foot.

Solution The height of the tree is the length of the perpendicular from the top of the tree to the ground. Use the ratio of sides of a right triangle.

We know the length of the side adjacent to the angle of elevation and we want to know the height of the tree, the length of the side opposite the angle of elevation.



$$\begin{aligned}\tan \theta &= \frac{\text{opp}}{\text{adj}} \\ \tan 57^\circ &= \frac{h}{12} \\ h &= 12 \tan 57^\circ \\ h &= 18.478\end{aligned}$$

Alternative Solution We know the measures of two angles and the included side. Find the measure of the third angle and use the Law of Sines.

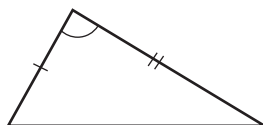
$$\begin{aligned}m\angle B &= 180 - (m\angle A + m\angle C) & \frac{a}{\sin A} &= \frac{b}{\sin B} \\ m\angle B &= 180 - (57 + 90) & \frac{h}{\sin 57^\circ} &= \frac{12}{\sin 33^\circ} \\ m\angle B &= 33 & h &= \frac{12 \sin 57^\circ}{\sin 33^\circ} \\ & & h &= 18.478\end{aligned}$$

Answer The tree is 18.5 feet tall. ■

The General Triangle

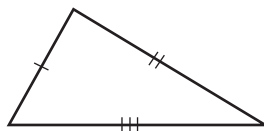
The Law of Cosines and the Law of Sines can be used to find the remaining three measures of any triangle when we know the measure of a side and the measures of any two other parts of the triangle.

CASE I *Given:* Two sides and the included angle

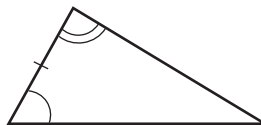


- Use the Law of Cosines to find the measure of the third side.

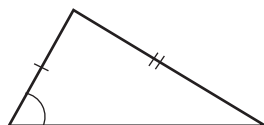
- Use the Law of Sines or the Law of Cosines to find the measure of another angle.
- Use the sum of the angles of a triangle to find the measure of the third angle.

CASE 2 *Given: Three sides*

- Use the Law of Cosines to find the measure of an angle.
- Use the Law of Sines or the Law of Cosines to find the measure of another angle.
- Use the sum of the angles of a triangle to find the measure of the third angle.

CASE 3 *Given: Two angles and a side*

- Use the sum of the angles of a triangle to find the measure of the third angle.
- Use the Law of Sines to find the remaining sides.

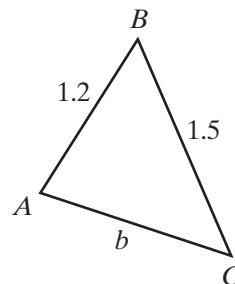
CASE 4 *Given: Two sides and an angle opposite one of them*

- Use the Law of Sines to find the possible measure(s) of another angle.
- Determine if there are two, one, or no possible triangles.
- If there is a triangle, use the sum of the angles of a triangle to find the measure(s) of the third angle.
- Use the Law of Sines or the Law of Cosines to find the measure(s) of the third side.

EXAMPLE 2

The Parks Department is laying out a nature trail through the woods that is to consist of three straight paths that form a triangle. The lengths of two paths measure 1.2 miles and 1.5 miles. What must be the measure of the angle between these two sections of the path in order that the total length of the nature trail will be 4.0 miles?

- Solution** (1) Draw a diagram. The trail forms a triangle, $\triangle ABC$, with $AB = c = 1.2$ and $BC = a = 1.5$:



- (2) The perimeter of the triangle is 4.0.
Find $CA = b$:

$$P = AB + BC + CA$$

$$4.0 = 1.2 + 1.5 + b$$

$$4.0 = 2.7 + b$$

$$1.3 = b$$

- (3) Use the Law of Cosines to find the measure of an angle when the measures of three sides are known. Find the measure of $\angle B$:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{1.5^2 + 1.2^2 - 1.3^2}{2(1.5)(1.2)}$$

$$\cos B = 0.5555 \dots$$

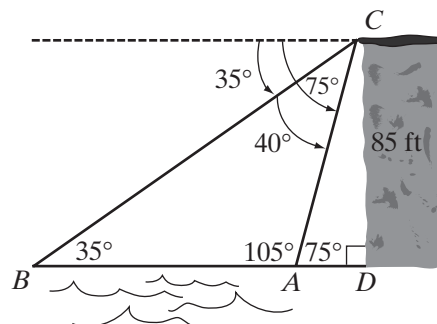
$$m\angle B = 56.25$$

Answer 56°

EXAMPLE 3

A man, standing on a cliff 85 feet high at the edge of the water, sees two ships. He positions himself so that the ships are in a straight line with a point directly below where he is standing. He estimates that the angle of depression of the closer ship is 75 degrees and the angle of depression of the farther ship is 35 degrees. How far apart are the two ships?

- Solution** (1) Draw and label a diagram. Let A be the closer ship, B the farther ship, C the edge of the cliff where the man is standing, and D the point directly below C at sea level. The angle of depression is the angle between a horizontal ray and a ray downward to the ship. Determine the measure of each angle in the diagram:



- (2) Use right $\triangle ACD$ to find AC , the measure of a side of both $\triangle ABC$ and $\triangle ACD$.

$$\sin 75^\circ = \frac{\text{opp}}{\text{hyp}}$$

$$\sin 75^\circ = \frac{85}{AC}$$

$$AC = \frac{85}{\sin 75^\circ}$$

$$AC \approx 87.998$$

- (3) In $\triangle ABC$, we now know the measures of two angles and the side opposite one of them. Use the Law of Sines to find AB , the required distance.

$$\frac{AB}{\sin \angle ACB} = \frac{AC}{\sin \angle ABC}$$

$$\frac{AB}{\sin 40^\circ} = \frac{87.998}{\sin 35^\circ}$$

$$AB = \frac{87.998 \sin 40^\circ}{\sin 35^\circ}$$

$$AB \approx 98.61$$

Answer The ships are 99 feet apart. ■

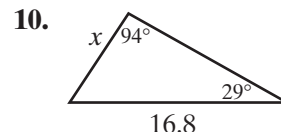
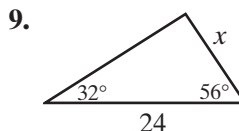
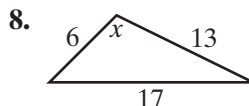
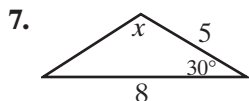
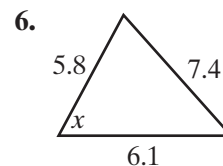
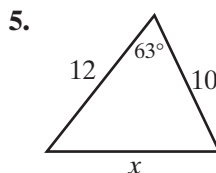
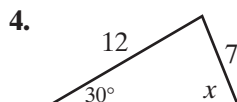
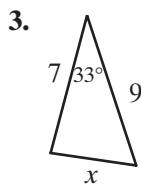
Exercises

Writing About Mathematics

- Navira said that in Example 2, it would have been possible to solve the problem by using $\triangle BDC$ to find BC first. Do you agree with Navira? Explain why or why not.
- Explain why an angle of depression is always congruent to an angle of elevation.

Developing Skills

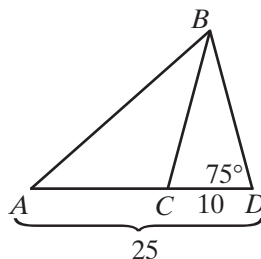
In 3–10: **a.** State whether each triangle should be solved by using the Law of Sines or the Law of Cosines. **b.** Solve each triangle, rounding answers to the nearest tenth. Include all possible solutions.



In 11–22, solve each triangle, that is, find the measures of the remaining three parts of the triangle to the nearest integer or the nearest degree.

- In $\triangle ABC$, $a = 15$, $b = 18$, and $m\angle C = 60$.
- In $\triangle ABC$, $a = 10$, $b = 12$, and $m\angle B = 30$.

13. In $\triangle ABC$, $b = 25$, $m\angle B = 45$, and $m\angle C = 60$.
14. In $\triangle ABC$, $a = 8$, $m\angle B = 35$, and $m\angle C = 55$.
15. In $\triangle DEF$, $d = 72$, $e = 48$, and $m\angle F = 110$.
16. In $\triangle PQR$, $p = 12$, $m\angle Q = 80$, and $m\angle R = 30$.
17. In $\triangle RST$, $r = 38$, $s = 28$, and $t = 18$.
18. In $\triangle ABC$, $a = 22$, $b = 18$, and $m\angle C = 130$.
19. In $\triangle PQR$, $p = 12$, $q = 16$, and $r = 20$.
20. In $\triangle DEF$, $d = 36$, $e = 72$, and $m\angle D = 30$.
21. In $\triangle RST$, $r = 15$, $s = 18$, and $m\angle T = 90$.
22. In $\triangle ABC$, $a = 15$, $b = 25$, and $c = 12$.
23. In the diagram, $AD = 25$, $CD = 10$, $BD = BC$, and $m\angle D = 75$. Find AB to the nearest tenth.



Applying Skills

24. A small park is in the shape of an isosceles trapezoid. The length of the longer of the parallel sides is 3.2 kilometers and the length of an adjacent side is 2.4 kilometers. A path from one corner of the park to an opposite corner is 3.6 kilometers long.
 - a. Find, to the nearest tenth, the measure of each angle between adjacent sides of the park.
 - b. Find, to the nearest tenth, the measure of each angle between the path and a side of the park.
 - c. Find, to the nearest tenth, the length of the shorter of the parallel sides.
25. From a point 50 feet from the foot of a vertical monument, the measure of the angle of elevation of the top of the monument is 65 degrees. What is the height of the monument to the nearest foot?
26. A vertical telephone pole that is 15 feet high is braced by two wires from the top of the pole to two points on the ground that are 5.0 feet apart on the same side of the pole and in a straight line with the foot of the pole. The shorter wire makes an angle of 65 degrees with the ground. Find the length of each wire to the nearest tenth.
27. From point C at the top of a cliff, two points, A and B , are sited on level ground. Points A and B are on a straight line with D , a point directly below C . The angle of depression of the nearer point, A , is 72 degrees and the angle of depression of the farther point, B , is 48 degree. If the points A and B are 20 feet apart, what is the height of the cliff to the nearest foot?
28. Mark is building a kite that is a quadrilateral with two pairs of congruent adjacent sides. One diagonal divides the kite into two unequal isosceles triangles and measures 14 inches. Each leg of one of the isosceles triangles measures 5 inches and each leg of the other measures 12 inches. Find the measures of the four angles of the quadrilateral.

CHAPTER SUMMARY

When we know the measures of three sides, two sides and the included angle, or two angles and any side of a triangle, the size and shape of the triangle are determined. We can use the Law of Cosines or the Law of Sines to find the measures of the remaining parts of the triangle.

$$\begin{array}{lll}
 \text{Law of Cosines:} & a^2 = b^2 + c^2 - 2bc \cos A & \cos A = \frac{b^2 + c^2 - a^2}{2bc} \\
 & b^2 = a^2 + c^2 - 2ac \cos B & \cos B = \frac{a^2 + c^2 - b^2}{2ac} \\
 & c^2 = a^2 + b^2 - 2ab \cos C & \cos C = \frac{a^2 + b^2 - c^2}{2ab}
 \end{array}$$

The Law of Cosines can be used to find the measure of the third side of a triangle when the measures of two sides and the included angle are known. The Law of Cosines can also be used to find the measure of any angle of a triangle when the measures of three sides are known.

$$\text{Law of Sines:} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The Law of Sines can be used to find the measure of a side of a triangle when the measures of any side and any two angles are known. The Law of Sines can also be used to find the number of possible triangles that can be drawn when the measures of two sides and an angle opposite one of them are known and can be used to determine the measures of the remaining side and angles if one or two triangles are possible.

$$\text{Area of a Triangle:} \quad \text{Area of } \triangle ABC = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C$$

The area of any triangle can be found if we know the measures of two sides and the included angle.

An **angle of elevation** is an angle between a horizontal line and a line that is rotated upward from the horizontal position. An **angle of depression** is an angle between a horizontal line and a line that is rotated downward from the horizontal position.

To determine the number of solutions given a , b , and $m\angle A$ in $\triangle ABC$, use the Law of Sines to solve for $\sin B$.

- If $\sin B > 1$, there is no triangle.
- If $\sin B = 1$, there is one right triangle if $\angle A$ is acute but no triangle if $\angle A$ is obtuse.
- If $\angle A$ is acute and $\sin B < 1$, find two possible values of $\angle B$:
 $0 < m\angle B < 90$ and $m\angle B' = 180 - m\angle B$.

If $m\angle A + m\angle B' < 180$, there are two possible triangles, $\triangle ABC$ and $\triangle AB'C$.

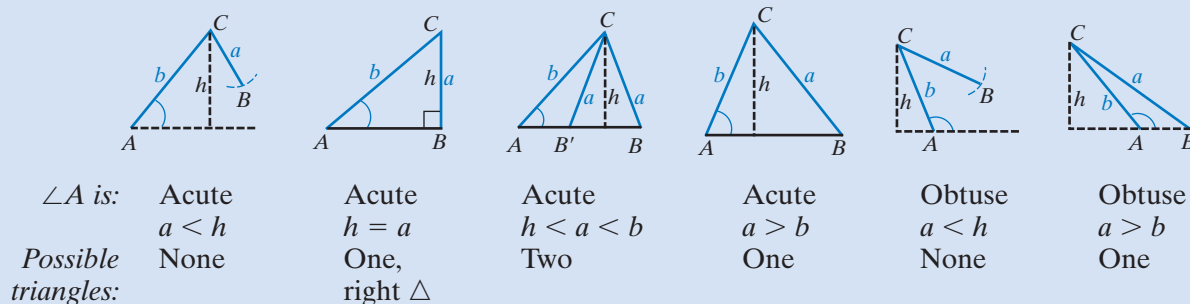
If $m\angle A + m\angle B' \geq 180$, $\triangle AB'C$ is not a triangle. There is only one possible triangle, $\triangle ABC$.

- If $\angle A$ is obtuse and $\sin B < 1$, $\angle B$ must be acute: $0 < m\angle B < 90$.

If $m\angle A + m\angle B < 180$, there is one triangle, $\triangle ABC$.

If $m\angle A + m\angle B \geq 180$, there is no triangle.

Alternatively, if we let $h = b \sin A$, the height of the triangle, we can summarize the number of possible triangles given a , b , and $m\angle A$ in $\triangle ABC$:



VOCABULARY

14-2 Law of Cosines

14-5 Law of Sines

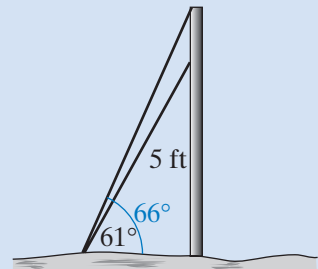
14-6 Ambiguous case

14-7 Angle of elevation • Angle of Depression

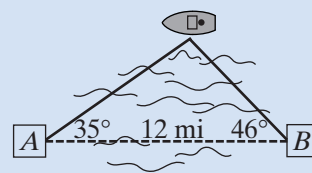
REVIEW EXERCISES

1. In $\triangle RST$, $RS = 18$, $RT = 27$, and $m\angle R = 50$. Find ST to the nearest integer.
2. The measures of two sides of a triangle are 12.0 inches and 15.0 inches. The measure of the angle included between these two sides is 80 degrees. Find the measure of the third side of the triangle to the nearest tenth of an inch.
3. In $\triangle DEF$, $DE = 84$, $EF = 76$, and $DF = 94$. Find, to the nearest degree, the measure of the smallest angle of the triangle.
4. The measures of three sides of a triangle are 22, 46, and 58. Find, to the nearest degree, the measure of the largest angle of the triangle.
5. Use the Law of Cosines to show that if the measures of the sides of a triangle are 10, 24, and 26, the triangle is a right triangle.

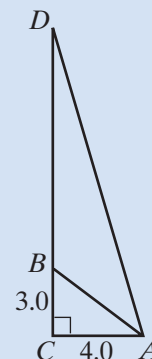
6. In $\triangle ABC$, $AB = 24$, $AC = 40$, and $m\angle A = 30$.
- Find the area of $\triangle ABC$.
 - Find the length of the altitude from C to \overline{AB} .
7. The lengths of the sides of a triangle are 8, 11, and 15.
- Find the measure of the smallest angle of the triangle.
 - Find the area of the triangle.
8. In $\triangle ABC$, $BC = 30.0$, $m\angle A = 70$, and $m\angle B = 55$. Find, to the nearest tenth, AB and AC .
9. The measures of two angles of a triangle are 100° and 46° . The length of the shortest sides of the triangle is 12. Find, to the nearest integer, the lengths of the other two sides.
10. In $\triangle ABC$, $a = 14$, $b = 16$, and $m\angle A = 48$.
- How many different triangles are possible?
 - Find the measures of $\angle B$ and of $\angle C$ if $\triangle ABC$ is an acute triangle.
 - Find the measures of $\angle B$ and of $\angle C$ if $\triangle ABC$ is an obtuse triangle.
11. Show that it is not possible to draw $\triangle PQR$ with $p = 12$, $r = 15$, and $m\angle P = 66$.
12. The measure of a side of a rhombus is 28.0 inches and the measure of the longer diagonal is 50.1 inches.
- Find, to the nearest degree, the measure of each angle of the rhombus.
 - Find, to the nearest tenth, the measure of the shorter diagonal.
 - Find, to the nearest integer, the area of the rhombus.
13. Use the Law of Cosines to find two possible lengths for AB of $\triangle ABC$ if $BC = 7$, $AC = 8$, and $m\angle A = 60$.
14. Use the Law of Sines to show that there are two possible triangles if $BC = 7$, $AC = 8$, and $m\angle A = 60$.
15. A vertical pole is braced by two wires that extend from different points on the pole to the same point on level ground. One wire is fastened to the pole 5.0 feet from the top of the pole and makes an angle of 61 degrees with the ground. The second wire is fastened to the top of the pole and makes an angle of 66 degrees with the ground. Find the height of the pole to the nearest tenth.



16. Coastguard station A is 12 miles west of coastguard station B along a straight coastline. A ship is sighted by the crew at station A to the northeast of the station at an angle of 35 degrees with the coastline and by the crew at station B to the northwest of the station at an angle of 46 degrees with the coastline. Find, to the nearest tenth, the distance from the ship to each of the stations.



17. In the diagram, $\triangle ABC$ is a right triangle with the right angle at C , $AC = 4.0$, $BC = 3.0$, and $m\angle BAC = \theta$. Side \overline{CB} is extended to D and $m\angle DAC = 2\theta$.
- Find the exact value of $\sin 2\theta$. (Hint: Use the double-angle formulas.)
 - Find θ to the nearest degree.
 - Find DC to the nearest tenth.



Exploration

Part A

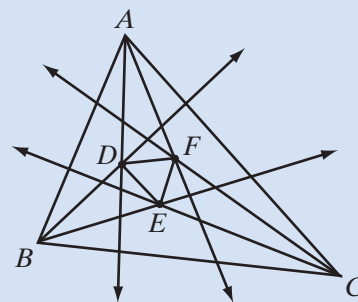
Use software to draw any triangle, $\triangle ABC$. Trisect each angle of the triangle.

Let D be the intersection of the trisection lines from A and B that are closest to \overline{AB} .

Let E be the intersection of the trisection lines from B and C that are closest to \overline{BC} .

Let F be the intersection of the trisection lines from C and A that are closest to \overline{CA} .

Measure \overline{DE} , \overline{EF} , and \overline{FD} . What seems to be true about $\triangle DEF$?



Part B

- Draw any triangle, $\triangle ABC$. Measure the lengths of the sides and the measures of the angles.
- Draw the trisection lines and label $\triangle DEF$ as in part A.
- Use the Law of Sines to find AD and BD in $\triangle ABD$.
- Use the Law of Sines to find BE and CE in $\triangle BCE$.

5. Use the Law of Sines to find CF and AF in $\triangle CAF$.
6. Use the Law of Cosines to find DF in $\triangle ADF$.
7. Use the Law of Cosines to find DE in $\triangle BDE$.
8. Use the Law of Cosines to find EF in $\triangle CEF$.
9. Is $\triangle DEF$ an equilateral triangle?

CUMULATIVE REVIEW**CHAPTERS 1–14****Part I**

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The sum of $\sqrt{-25}$ and $2\sqrt{-16}$ is
 (1) -13 (2) $13i$ (3) -18 (4) $18i$
2. If $f(\theta) = \cot \theta$, then $f\left(\frac{\pi}{2}\right)$ is
 (1) 0 (2) 1 (3) ∞ (4) undefined
3. The expression $\sum_{k=1}^3 (k + 1)^2$ is equal to
 (1) 13 (2) 14 (3) 18 (4) 29
4. If $f(x) = x - 1$ and $g(x) = x^2$, then $f(g(3))$ is equal to
 (1) 18 (2) 11 (3) 8 (4) 4
5. The sum $\sin 50^\circ \cos 30^\circ + \cos 50^\circ \sin 30^\circ$ is equal to
 (1) $\sin 80^\circ$ (2) $\sin 20^\circ$ (3) $\cos 80^\circ$ (4) $\cos 20^\circ$
6. If $\log_x \frac{1}{4} = -2$, then x equals
 (1) 16 (2) 2 (3) $\frac{1}{2}$ (4) $\frac{1}{16}$
7. The solution set of $x^2 - 2x + 5 = 0$ is
 (1) $\{-1, 3\}$ (3) $\{1 - 2i, 1 + 2i\}$
 (2) $\{-3, 1\}$ (4) $\{-1 - 2i, 1 + 2i\}$
8. In simplest form, the fraction $\frac{\frac{a}{b} - \frac{b}{a}}{\frac{1}{a} - \frac{1}{b}}$ is equal to
 (1) $a + b$ (2) $a - b$ (3) $-(a + b)$ (4) $b - a$
9. An angle of $\frac{7\pi}{4}$ radians is congruent to an angle of
 (1) 135° (2) 225° (3) 315° (4) 405°

10. Which of the following is a geometric sequence?

(1) 1, 3, 5, 7, ...

(3) 1, 0.1, 0.01, 0.001, ...

(2) 1, 3, 6, 10, ...

(4) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Use the exact values of the $\sin 30^\circ$, $\sin 45^\circ$, $\cos 30^\circ$, and $\cos 45^\circ$ to find the exact value of $\cos 15^\circ$.

12. Write the fraction $\frac{1 - \sqrt{5}}{1 + \sqrt{5}}$ as an equivalent fraction with a rational denominator.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Solve for x and check: $3 - \sqrt{2x - 3} = x$

14. Find all values of x in the interval $0^\circ \leq x < 360^\circ$ that satisfy the following equation:

$$6 \sin^2 x - 5 \sin x - 4 = 0$$

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. If $\log_b x = \log_b 3 + 2 \log_b 4 - \frac{1}{2} \log_b 8$, express x in simplest form.

16. a. Sketch the graph of $y = 2 \sin x$ in the interval $0 \leq x \leq 2\pi$.

b. On the same set of axes, sketch the graph of $y = \cos 2x$.

c. For how many values of x in the interval $0 \leq x \leq 2\pi$ does $2 \sin x = \cos 2x$?