CHAPTER

8

LOGARITHMIC FUNCTIONS

The heavenly bodies have always fascinated and challenged humankind. Our earliest records contain conclusions, some false and some true, that were believed about the relationships among the sun, the moon, Earth, and the other planets. As more accurate instruments for studying the heavens became available and more accurate measurements were possible, the mathematical computations absorbed a great amount of the astronomer's time.

A basic principle of mathematical computation is that it is easier to add than to multiply. John Napier (1550–1617) developed a system of logarithms that facilitated computation by using the principles of exponents to find a product by using addition. Henry Briggs (1560–1630) developed Napier's concept using base 10. Seldom has a new mathematical concept been more quickly accepted.

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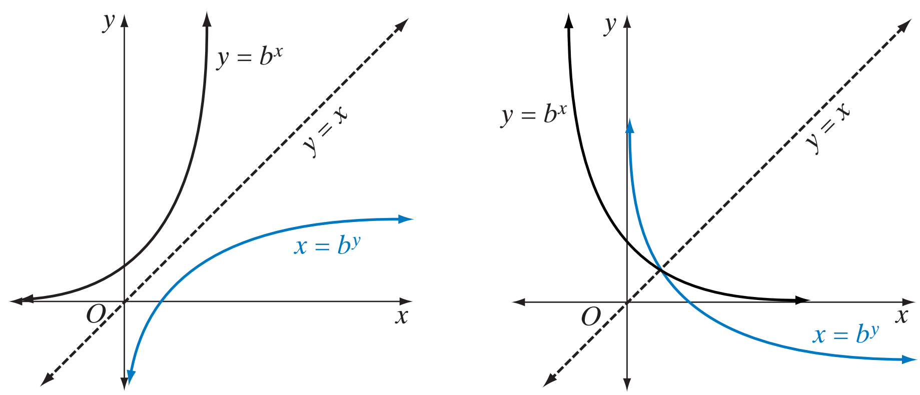
8-1 INVERSE OF AN EXPONENTIAL FUNCTION

In Chapter 7, we showed that any positive real number can be the exponent of a power by drawing the graph of the exponential function $y = b^x$ for $0 < b < 1$ or $b > 1$. Since $y = b^x$ is a one-to-one function, its reflection in the line $y = x$ is also a function. The function $x = b^y$ is the inverse function of $y = b^x$.



$b > 1$ $\qquad$ $0 < b < 1$

The equation of a function is usually solved for y in terms of x. To solve the equation $x = b^y$ for y, we need to introduce some new terminology. First we will describe y in words:

$x = b^y$: "y is the exponent to the base b such that the power is x."

A **logarithm** is an exponent. Therefore, we can write:

$x = b^y$: "y is the *logarithm* to the base b of the power x."

The word *logarithm* is abbreviated as *log*. Look at the essential parts of this sentence:

$y = \log_b x$: "***y* is** the **log**arithm to the **base *b*** of ***x***."

The base b is written as a subscript to the word "log."

► **$x = b^y$ can be written as $y = \log_b x$.**

For example, let $b = 2$. Write pairs of values for $x = 2^y$ and $y = \log_2 x$.

$x = 2^y$	In Words	$y = \log_2 x$	(x, y)
$\frac{1}{2} = 2^{-1}$	-1 is the logarithm to the base 2 of $\frac{1}{2}$.	$-1 = \log_2 \frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
$1 = 2^0$	0 is the logarithm to the base 2 of 1.	$0 = \log_2 1$	$(1, 0)$
$\sqrt{2} = 2^{\frac{1}{2}}$	$\frac{1}{2}$ is the logarithm to the base 2 of $\sqrt{2}$.	$\frac{1}{2} = \log_2 \sqrt{2}$	$\left(\sqrt{2}, \frac{1}{2}\right)$
$2 = 2^1$	1 is the logarithm to the base 2 of 2.	$1 = \log_2 2$	$(2, 1)$
$4 = 2^2$	2 is the logarithm to the base 2 of 4.	$2 = \log_2 4$	$(4, 2)$
$8 = 2^3$	3 is the logarithm to the base 2 of 8.	$3 = \log_2 8$	$(8, 3)$

We say that $y = \log_b x$, with b a positive number not equal to 1, is a **logarithmic function**.

EXAMPLE 1

Write the equation $x = 10^y$ for y in terms of x.

Solution $x = 10^y \leftarrow$ y is the exponent or logarithm to the base 10 of x.

$y = \log_{10} x$

Graphs of Logarithmic Functions

From our study of exponential functions in Chapter 7, we know that when $b > 1$ and when $0 < b < 1$, $y = b^x$ is defined for all real values of x. Therefore, the domain of $y = b^x$ is the set of real numbers. When $b > 1$, as the negative values of x get larger and larger in absolute value, the value of b^x gets smaller but is always positive. When $0 < b < 1$, as the positive values of x get larger and larger, the value of b^x gets smaller but is always positive. Therefore, the range of $y = b^x$ is the set of positive real numbers.

When we interchange x and y to form the inverse function $x = b^y$ or $y = \log_b x$:

- **The domain of $y = \log_b x$ is the set of positive real numbers.**
- **The range $y = \log_b x$ is the set of real numbers.**
- **The y-axis or the line $x = 0$ is a *vertical asymptote* of $y = \log_b x$.**

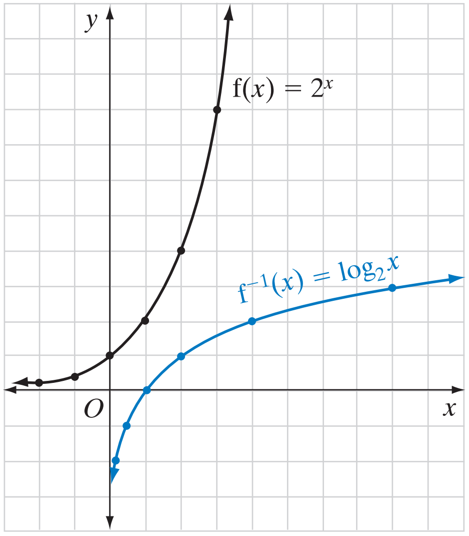
EXAMPLE 2

a. Sketch the graph of $f(x) = 2^x$.

b. Write the equation of $f^{-1}(x)$ and sketch its graph.

Solution **a.** Make a table of values for $f(x) = 2^x$, plot the points, and draw the curve.

x	2^x	$f(x)$
-2	$2^{-2} = \frac{1}{2^2}$	$\frac{1}{4}$
-1	$2^{-1} = \frac{1}{2}$	$\frac{1}{2}$
0	2^0	1
1	2^1	2
2	2^2	4
3	2^3	8



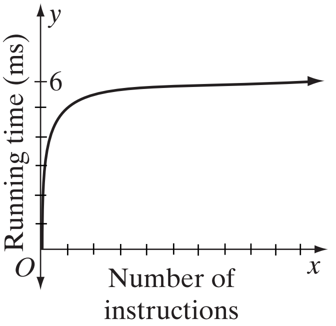
b. Let $f(x) = 2^x \rightarrow y = 2^x$.

To write $f^{-1}(x)$, interchange x and y.

$x = 2^y$ is written as $y = \log_2 x$. Therefore, $f^{-1}(x) = \log_2 x$.

To draw the graph, interchange x and y in each ordered pair or reflect the graph of $f(x)$ over the line $y = x$. Ordered pairs of $f^{-1}(x)$ include $\left(\frac{1}{4}, -2\right)$, $\left(\frac{1}{2}, -1\right)$, $(1, 0)$, $(2, 1)$, $(4, 2)$, and $(8, 3)$.

The function $y = \log_b x$ represents *logarithmic growth.* Quantities represented by a logarithmic function grow very slowly. For example, suppose that the time it takes for a computer to run a program could by modeled by the logarithmic function $y = \log_{10} x$ where x is the number of instructions of the computer program and y is the running time in milliseconds. The graph on the right shows the running time in the interval $1 < x < 1{,}000{,}000$. Note that the graph increases from 0 to 5 for the first 100,000 instructions but only increases from 5 to 6 as the number of instructions increases from 100,000 to 1,000,000. (Each interval on the x-axis represents 100,000 instructions.)



Exercises

Writing About Mathematics

1. Peg said that $(1, 0)$ is always a point on the graph of $y = \log_b x$. Do you agree with Peg? Explain why or why not.

2. Sue said that if $x = b^{2y}$ for $b > 1$, then $y = \frac{1}{2}\log_b x$. Do you agree with Sue? Explain why or why not.

Developing Skills

In 3–10: **a.** For each f(x), write an equation for $f^{-1}(x)$, the inverse function. **b.** Sketch the graph of f(x) and of $f^{-1}(x)$.

3. $f(x) = 3^x$ **4.** $f(x) = 5^x$ **5.** $f(x) = \left(\frac{3}{2}\right)^x$ **6.** $f(x) = \left(\frac{5}{2}\right)^x$

7. $f(x) = \left(\frac{1}{2}\right)^x$ **8.** $f(x) = \left(\sqrt{2}\right)^x$ **9.** $f(x) = \frac{1}{3^x}$ **10.** $f(x) = -2^x$

In 11–22, solve each equation for y in terms of x.

11. $x = 6^y$ **12.** $x = 10^y$ **13.** $x = 8^y$ **14.** $x = (0.1)^y$

15. $x = (0.2)^y$ **16.** $x = 4^{-y}$ **17.** $x = 12^{-y}$ **18.** $x = \log_2 y$

19. $x = \log_5 y$ **20.** $x = \log_{10} y$ **21.** $x = \log_8 y$ **22.** $x = \log_{0.1} y$

Applying Skills

23. Genevieve decided to organize a group of volunteers to help at a soup kitchen. Every week for the first three weeks, the number of volunteers tripled so that the number, f(x), after x weeks is $f(x) = 3^x$.

a. Write the ordered pairs of the function $f(x) = 3^x$ for $0 \le x \le 3$ and locate the pairs as points on a graph. The domain is the set of non-negative integers.

b. Write the ordered pairs for $f^{-1}(x)$ and sketch the graph.

24. If money is invested at a rate of 5% compounded annually, then for each dollar invested, the amount of money in an account is g(x), when $g(x) = 1.05^x$ after x years.

a. Write the ordered pairs of the function g for $0 \le x \le 3$ and locate the pairs as points on a graph. The domain is the set of non-negative integers.

b. Write the ordered pairs for $g^{-1}(x)$ and sketch the graph.

8-2 LOGARITHMIC FORM OF AN EXPONENTIAL EQUATION

An exponential equation and a logarithmic equation are two different ways of expressing the same relationship. For example, to change an exponential equation to a logarithmic equation, recall that a logarithm or log is an exponent. In the exponential equation $81 = 3^4$, the exponent or log is 4. The basic statement is:

$$\log = 4$$

Then write the base as a subscript of the word "log" to indicate the log to the base 3:

$$\log_3 = 4$$

Now add the value of the power, that is, log to the base 3 of the power 81:

$$\log_3 81 = 4$$

To change from a logarithmic equation to an exponential statement, we must again look for the exponent. For example, in the logarithmic equation $\log_{10} 0.01 = -2$, the basic statement is that the log or exponent is -2. The base is the number that is written as a subscript of the word "log":

$$10^{-2}$$

The number that follows the word "log" is the power:

$$10^{-2} = 0.01$$

Recall that $10^{-2} = \frac{1}{10^2} = \frac{1}{100} = 0.01$.

In general:

► **$b^a = c$ is equivalent to $\log_b c = a$.**

Exponent Power

$$b^a = c$$

Base

Power Exponent

$$\log_b c = a$$

Base

EXAMPLE 1

Write $9^3 = 729$ in logarithmic form and express its meaning in words.

Solution In the equation $9^3 = 729$, the logarithm (exponent) is 3:

$$\log = 3$$

Write the base, 9, as a subscript of "log":

$$\log_9 = 3$$

Write the power, 729, after "log":

$$\log_9 729 = 3$$

The logarithmic form of the equation means, "the exponent to the base 9 of the power 729 is 3." *Answer*

EXAMPLE 2

Write $\frac{3}{2} = \log_{25} 125$ in exponential form.

Solution The equation $\frac{3}{2} = \log_{25} 125$ says that $\frac{3}{2}$ is the exponent. The base is written as the subscript of "log." The power, $25^{\frac{3}{2}}$, is 125.

$$25^{\frac{3}{2}} = 125 \quad \textit{Answer}$$

Note that $25^{\frac{3}{2}} = \left(25^{\frac{1}{2}}\right)^3 = \left(\sqrt{25}\right)^3 = 5^3 = 125.$

EXAMPLE 3

Find the value of a when $\log_8 a = \frac{1}{3}$.

Solution The equation $\log_8 a = \frac{1}{3}$ means that for the power a, the base is 8, and the exponent is $\frac{1}{3}$.

$$8^{\frac{1}{3}} = a$$
$$\sqrt[3]{8} = a$$
$$2 = a \quad \textit{Answer}$$

EXAMPLE 4

Solve the following equation for x: $\log_3 \frac{1}{9} = \log_2 x$.

Solution Represent each side of the equation by y.

Let $y = \log_3 \frac{1}{9}$.	Let $y = \log_2 x$.
Then:	Then:
$3^y = \frac{1}{9}$	$-2 = \log_2 x$
$3^y = 3^{-2}$	$2^{-2} = x$
$y = -2$	$\frac{1}{4} = x$

Answer $x = \frac{1}{4}$

Exercises

Writing About Mathematics

1. If $\log_b c = a$, explain why $\log_b \frac{1}{c} = -a$.
2. If $\log_b c = a$, explain why $\log_b c^2 = 2a$.

Developing Skills

In 3–14, write each exponential equation in logarithmic form.

3. $2^4 = 16$
4. $5^3 = 125$
5. $64 = 8^2$
6. $12^0 = 1$
7. $216 = 6^3$
8. $10^{-1} = 0.1$
9. $5^{-3} = 0.008$
10. $4^{-2} = 0.0625$
11. $7^{-1} = \frac{1}{7}$
12. $64^{\frac{1}{3}} = 4$
13. $625^{\frac{3}{4}} = 125$
14. $0.001 = 100^{-\frac{3}{2}}$

In 15–26, write each logarithmic equation in exponential form.

15. $\log_{10} 100 = 2$
16. $\log_5 125 = 3$
17. $\log_4 16 = 2$
18. $7 = \log_2 128$
19. $5 = \log_3 243$
20. $\log_7 1 = 0$
21. $\log_{10} 0.001 = -3$
22. $\log_{100} 0.01 = -1$
23. $-2 = \log_5 0.04$
24. $\log_8 2 = \frac{1}{3}$
25. $\log_{49} 343 = \frac{3}{2}$
26. $-\frac{2}{5} = \log_{32} 0.25$

In 27–56, evaluate each logarithmic expression. Show all work.

27. $\log_8 8$
28. $5 \log_8 8$
29. $\log_6 216$
30. $4 \log_6 216$
31. $\log_{\frac{1}{2}} 2$
32. $8 \log_{\frac{1}{2}} 2$
33. $\log_4 \frac{1}{16}$
34. $3 \log_4 \frac{1}{16}$
35. $\log_3 729$
36. $\frac{1}{3} \log_3 729$
37. $\log_4 \frac{1}{64}$
38. $16 \log_4 \frac{1}{64}$
39. $\log_3 81$
40. $\log_2 16$
41. $\log_3 81 \cdot \log_2 16$
42. $\log_5 125$
43. $\log_{10} 1{,}000$
44. $\log_2 32$
45. $\log_{\frac{1}{2}} \frac{1}{4}$
46. $\log_3 81$
47. $\log_{18} 324$
48. $\log_6 36$
49. $\log_{\frac{1}{3}} 27$
50. $\dfrac{6 \log_6 36}{\log_{\frac{1}{3}} 27}$
51. $\log_5 125 \cdot \log_{10} 1{,}000 \cdot 2 \log_2 32$
52. $\log_{\frac{1}{2}} \frac{1}{4} \cdot \log_3 81 \cdot \frac{1}{2} \log_{18} 324$

53. $\dfrac{\log_5 25 + 2\log_{10} 10}{\log_{16} 4}$

54. $\dfrac{2\log_{1.5} 2.25}{\log_4 64 - \log_{80} 80}$

55. $\dfrac{3\log_3 9 \cdot 4\log_8 8 \cdot \log_{13} 169}{6\log_2 256 + \log_{\frac{1}{2}} 8}$

56. $\dfrac{\log_3 27 + 8\log_{16} 2}{\log_8 512} \cdot \log_{1{,}000} 10$

In 57–68, solve each equation for the variable.

57. $\log_{10} x = 3$

58. $\log_2 32 = x$

59. $\log_5 625 = x$

60. $a = \log_4 16$

61. $\log_b 27 = 3$

62. $\log_b 64 = 6$

63. $\log_5 y = -2$

64. $\log_{25} c = -4$

65. $\log_{100} x = -\frac{1}{2}$

66. $\log_8 x = \frac{1}{2}$

67. $\log_{36} 6 = x$

68. $\log_b 1{,}000 = \frac{3}{2}$

69. If $f(x) = \log_3 x$, find f(81).

70. If $p(x) = \log_{25} x$, find p(5).

71. If $g(x) = \log_{10} x$, find g(0.001).

72. If $h(x) = \log_{32} x$, find h(8).

73. Solve for a: $\log_5 0.2 = \log_a 10$

74. Solve for x: $\log_{100} 10 = \log_{16} x$

Applying Skills

75. When \$1 is invested at 6% interest, its value, A, after t years is $A = 1.06^t$. Express t in terms of A.

76. R is the ratio of the population of a town n years from now to the population now. If the population has been decreasing by 3% each year, $R = 0.97^n$. Express n in terms of R.

77. The decay constant of radium is -0.0004 per year. The amount of radium, A, present after t years in a sample that originally contained 1 gram of radium is $A = e^{-0.0004t}$.

a. Express $-0.0004t$ in terms of A and e.

b. Solve for t in terms of A.

8-3 LOGARITHMIC RELATIONSHIPS

Because a logarithm is an exponent, the rules for exponents can be used to derive the rules for logarithms. Just as the rules for exponents only apply to powers with like bases, the rules for logarithms will apply to logarithms with the same base.

Basic Properties of Logarithms

If $0 < b < 1$ or $b > 1$:

$$b^0 = 1 \leftrightarrow \log_b 1 = 0$$

$$b^1 = b \leftrightarrow \log_b b = 1$$

For example:

$$3^0 = 1 \leftrightarrow \log_3 1 = 1$$
$$3^1 = 3 \leftrightarrow \log_3 3 = 1$$

Logarithms of Products

If $0 < b < 1$ or $b > 1$:

$$b^x = c \leftrightarrow \log_b c = x$$
$$b^y = d \leftrightarrow \log_b d = y$$
$$b^{x+y} = cd \leftrightarrow \log_b cd = (x + y)$$

Therefore,

$$\mathbf{\log_b cd = \log_b c + \log_b d}$$

► **The log of a product is the sum of the logs of the factors of the product.**

For example:

$$3^2 = 9 \leftrightarrow \log_3 9 = 2$$
$$3^4 = 81 \leftrightarrow \log_3 81 = 4$$
$$3^{2+4} = 9 \times 81 \leftrightarrow \log_3 (9 \times 81) = \log_3 9 + \log_3 81 = 2 + 4 = 6$$

EXAMPLE 1

If $\log_5 125 = 3$ and $\log_5 25 = 2$, find $\log_5 (125 \times 25)$.

Solution

$$\log_b cd = \log_b c + \log_b d$$
$$\log_5 (125 \times 25) = \log_5 125 + \log_5 25$$
$$\log_5 (125 \times 25) = 3 + 2$$
$$\log_5 (125 \times 25) = 5 \quad \textit{Answer}$$

Logarithms of Quotients

If $0 < b < 1$ or $b > 1$:

$$b^x = c \leftrightarrow \log_b c = x$$
$$b^y = d \leftrightarrow \log_b d = y$$
$$b^{x-y} = \frac{c}{d} \leftrightarrow \log_b \frac{c}{d} = (x - y)$$

Therefore,

$$\log_b \frac{c}{d} = \log_b c - \log_b d$$

► **The log of a quotient is the log of the dividend minus the log of the divisor.**

For example:

$$3^2 = 9 \leftrightarrow \log_3 9 = 2$$
$$3^4 = 81 \leftrightarrow \log_3 81 = 4$$
$$3^{2-4} = \frac{9}{81} \leftrightarrow \log_3\left(\frac{9}{81}\right) = \log_3 9 - \log_3 81 = 2 - 4 = -2$$

Do we get the same answer if the quotient is simplified?

$$\frac{9}{81} = \frac{1}{9}$$
$$\log_3 1 = 0 \qquad \log_3 9 = 2$$
$$\log_3\left(\frac{1}{9}\right) = \log_3 1 - \log_3 9 = 0 - 2 = -2 \checkmark$$

This is true because $3^{2-4} = 3^{-2} = \frac{1}{9}$.

EXAMPLE 2

Use logs to show that for $b > 0$, $b^0 = 1$.

Solution Let $\log_b c = a$. Then,

$$\log_b c - \log_b c = \log_b\left(\frac{c}{c}\right)$$
$$a - a = \log_b 1$$
$$0 = \log_b 1$$
$$b^0 = 1$$

Logarithms of Powers

If $0 < b < 1$ or $b > 1$:

$$b^x = c \leftrightarrow \log_b c = x$$
$$(b^x)^a = b^{ax} = c^a \leftrightarrow \log_b c^a = ax$$

Therefore,

$$\log_b c^a = a \log_b c$$

► **The log of a power is the exponent times the log of the base.**

For example:

$$3^2 = 9 \leftrightarrow \log_3 9 = 2$$
$$(3^2)^3 = 3^{2\times 3} = 9^3 \leftrightarrow \log_3 (9)^3 = 3 \log_3 9 = 3(2) = 6$$

EXAMPLE 3

Find $\log_2 8$: **a.** using the product rule, and **b.** using the power rule.

Solution We know that $\log_2 2 = 1$.

a. Product rule:
$$\begin{aligned}\log_2 8 = \log_2 (2 \times 2 \times 2) &= \log_2 2 + \log_2 2 + \log_2 2\\ &= 1 + 1 + 1\\ &= 3\end{aligned}$$

b. Power rule:
$$\begin{aligned}\log_2 8 = \log_2 (2^3) &= 3 \log_2 2\\ &= 3(1)\\ &= 3\end{aligned}$$

Check $2^3 = 8 \leftrightarrow \log_2 8 = 3$

Answer Using both the product rule and the power rule, $\log_2 8 = 3$.

EXAMPLE 4

Find the value of $\log_{12} 9 + \log_{12} 16$.

Solution Use the rule for product of logs:

$$\log_{12} 9 + \log_{12} 16 = \log_{12} (9 \times 16) = \log_{12} 144$$

Since $12^2 = 144$, $\log_{12} 144 = 2$.

Answer $\log_{12} 9 + \log_{12} 16 = 2$

EXAMPLE 5

If $\log_{10} 3 = x$ and $\log_{10} 5 = y$, express $\log_{10} \frac{\sqrt{3}}{5^2}$ in terms of x and y.

Solution

$$\begin{aligned}\log_{10} \frac{\sqrt{3}}{5^2} = \log_{10} \frac{3^{\frac{1}{2}}}{5^2} &= \log_{10} 3^{\frac{1}{2}} - \log_{10} 5^2\\ &= \tfrac{1}{2} \log_{10} 3 - 2 \log_{10} 5\\ &= \tfrac{1}{2}x - 2y\end{aligned}$$

Answer $\log_{10} \frac{\sqrt{3}}{5^2} = \frac{1}{2}x - 2y$

SUMMARY

	Logarithms
Multiplication	$\log_b cd = \log_b c + \log_b d$
Division	$\log_b \frac{c}{d} = \log_b c - \log_b d$
Logarithm of a Power	$\log_b c^a = a \log_b c$
Logarithm of 1	$\log_b 1 = 0$
Logarithm of the Base	$\log_b b = 1$

Exercises

Writing About Mathematics

1. Show that for all $a > 0$ and $a \neq 1$, $\log_a a^n = n$.

2. Terence said that $(\log_a b) \cdot (\log_a c) = \log_a bc$. Do you agree with Terence? Explain why or why not.

Developing Skills

In 3–14, use the table to the right and the properties of logarithms to evaluate each expression. Show all work.

$\log_3 \frac{1}{9} = -2$	$\log_3 243 = 5$
$\log_3 \frac{1}{3} = -1$	$\log_3 729 = 6$
$\log_3 1 = 0$	$\log_3 2{,}187 = 7$
$\log_3 3 = 1$	$\log_3 6{,}561 = 8$
$\log_3 9 = 2$	$\log_3 19{,}683 = 9$
$\log_3 27 = 3$	$\log_3 59{,}049 = 10$
$\log_3 81 = 4$	

3. 27×81

4. 243×27

5. $19{,}683 \div 729$

6. $6{,}561 \div 27$

7. 9^4

8. 243^2

9. $81^2 \times 9$

10. $\sqrt{6{,}561} \div 729$

11. $\sqrt[4]{243 \times 2{,}187}$

12. $\sqrt{\frac{19{,}683}{2{,}187}}$

13. $81^3 \div \sqrt{729}$

14. $27 \times \sqrt[3]{\frac{729}{19{,}683}}$

In 15–20: **a.** Write each expression as a single logarithm. **b.** Find the value of each expression.

15. $\log_3 1 + \log_3 9$

16. $\log_3 27 + \log_3 81$

17. $\log_3 243 - \log_3 729$

18. $\log_3 6{,}561 - \log_3 243$

19. $\frac{1}{3} \log_3 2{,}187 + \frac{1}{6} \log_3 81$

20. $\log_3 9 - 2 \log_3 27 + \log_3 243$

In 21–23: **a.** Expand each expression as a difference, sum, and/or multiple of logarithms. **b.** Find the value of each expression.

21. $4 \log_3 \frac{9}{27}$

22. $\frac{1}{2} \log_3 3(243)$

23. $\log_4 \sqrt{16^2}$

In 24–29, write each expression as a single logarithm.

24. $\log_e x + \log_e 10$

25. $\log_2 a + \log_2 b$

26. $4 \log_2 (x + 2)$

27. $\log_{10} y - 2 \log_{10} (y - 1)$

28. $\log_e x + 2 \log_e y - 2 \log_e z$

29. $\frac{1}{2} \log_3 x^{10} - \frac{2}{5} \log_3 x^5$

In 30–35, expand each expression using the properties of logarithms.

30. $\log_2 2ab$

31. $\log_3 \frac{10}{x}$

32. $\log_5 a^{-5}$

33. $\log_{10} (x + 1)^2$

34. $\log_4 \frac{x^6}{y^5}$

35. $\log_e \sqrt{x}$

In 36–47, write each expression in terms of A and B if $\log_2 x = A$ and $\log_2 y = B$.

36. $\log_2 xy$

37. $\log_2 x^2y$

38. $\log_2 (xy)^3$

39. $\log_2 xy^3$

40. $\log_2 (x \div y)$

41. $\log_2 (x^2 \div y^3)$

42. $\log_2 \sqrt{xy}$

43. $\log_2 x\sqrt{y}$

44. $\log_2 \frac{\sqrt{x}}{y^3}$

45. $\log_2 \sqrt{\frac{x}{y}}$

46. $\log_2 x\sqrt{x}$

47. $\log_2 \sqrt[4]{y}$

In 48–53, solve each equation for the variable.

48. $\log_2 2^3 + \log_2 2^2 = \log_2 x$

49. $\log_2 16 + \log_2 2 = \log_2 x$

50. $\log_2 x - \log_2 8 = \log_2 4$

51. $\log_5 x + \log_5 x = \log_5 625$

52. $\log_b 64 - \log_b 16 = \log_4 16$

53. $\log_2 8 + \log_3 9 = \log_b 100{,}000$

8-4 COMMON LOGARITHMS

DEFINITION

A **common logarithm** is a logarithm to the base 10.

When the base is 10, the base need not be written as a subscript of the word "log."

$$\log_{10} 100 = \log 100$$

Before calculators, tables of common logarithms were used for computation. The slide rule, a mechanical device used for multiplication and division, was a common tool for engineers and scientists. Although logarithms are no longer needed for ordinary arithmetic computations, they make it possible for us to solve equations, particularly those in which the variable is an exponent.

The TI-83+/84+ graphing calculator has a LOG key that will return the common logarithm of any number. For example, to find $\log_{10} 8$ or $\log 8$, enter the following sequence of keys.

ENTER: LOG 8 ENTER

DISPLAY:
```
LOG(8
          .903089987
```

Therefore, $\log_{10} 8 \approx 0.903089987$ or $10^{0.903089987} \approx 8$.

To show that this last statement is true, store the value of log 8 as x, and then use 10^x, the 2nd function of the LOG key.

ENTER: STO▶ X,T,Θ,n ENTER 2nd 10^x X,T,Θ,n ENTER

DISPLAY:
```
Ans→X
          .903089987
10^(X
                   8
```

In the equation $10^{0.903089987} = 8$, the number 0.903089987 is the logarithm of 8 and 8 is the **antilogarithm** of 0.903089987. The logarithm is the exponent and the antilogarithm is the power. In general:

► **For $\log_b x = y$, x is the antilogarithm of y.**

EXAMPLE 1

Find log 23.75.

Solution Use the LOG key of the calculator.

ENTER: LOG 23.75 ENTER

DISPLAY:
```
LOG(23.75
          1.375663614
```

Answer $\log 23.75 \approx 1.375663614$

EXAMPLE 2

If $\log x = 2.87534$, find x to the nearest tenth.

Solution Log $x = 2.87534$ can be written as $10^{2.87534} = x$.

Use the **10^x** key to find the antilogarithm.

ENTER: **2nd** **10^x** 2.87534 **ENTER**

DISPLAY:

```
10^(2.87534
         750.4815156
```

Answer To the nearest tenth, $\log 750.5 = 2.87534$ or $10^{2.87534} = 750.5$.

EXAMPLE 3

Use logarithms and antilogarithms to find the value of $\frac{4.20^3 \times 0.781}{4.83}$ to the nearest tenth.

Solution First take the log of the expression.

$$\log \frac{4.20^3 \times 0.781}{4.83} = 3 \log 4.20 + \log 0.781 - \log 4.83$$

ENTER: 3 **LOG** 4.2 **)** **+** **LOG** .781 **)** **−** **LOG** 4.83 **ENTER**

DISPLAY:

```
3log(4.2)+log(.7
81)-log(4.83
         1.078451774
```

Now store this value for x and find its antilogarithm.

ENTER: **STO▸** **X,T,Θ,n** **ENTER** **2nd** **10^x** **X,T,Θ,n** **ENTER**

DISPLAY:

```
Ans→X
         1.078451774
10^(X
         11.97986087
```

Answer To the nearest tenth, $\frac{4.20^3 \times 0.781}{4.83} = 12.0$.

Exercises

Writing About Mathematics

1. Explain why $\log 80 = 1 + \log 8$.
2. Explain why $\log x$ is negative if $0 < x < 1$.

Developing Skills

In 3–14, find the common logarithm of each number to the nearest hundredth.

3. 3.75 **4.** 8.56 **5.** 47.88 **6.** 56.2
7. 562 **8.** 5,620 **9.** 0.342 **10.** 0.0759
11. 1 **12.** 10 **13.** 100 **14.** 0.1

In 15–23, evaluate each logarithm to the nearest hundredth.

15. $\log 1{,}024$ **16.** $\log 80$ **17.** $\log 0.002$
18. $\log 9 + \log 3$ **19.** $\log 64 - \log 16$ **20.** $200 \log \frac{5}{2}$
21. $\frac{3 \log 4}{\log 5}$ **22.** $\frac{\log 100 - \frac{1}{2}\log 36}{\log 6}$ **23.** $\log \frac{1}{2} \cdot \log 100 \cdot \log 300$

In 24–35, for each given logarithm, find the antilogarithm, x. Write the answer to four decimal places.

24. $\log x = 0.5787$ **25.** $\log x = 0.8297$ **26.** $\log x = 1.3826$
27. $\log x = 1.7790$ **28.** $\log x = 2.2030$ **29.** $\log x = 2.5619$
30. $\log x = 4.8200$ **31.** $\log x = -0.5373$ **32.** $\log x = -0.05729$
33. $\log x = -1.1544$ **34.** $\log x = -3$ **35.** $\log x = -4$

In 36–47, if $\log 3 = x$ and $\log 5 = y$, write each of the logs in terms of x and y.

36. $\log 15$ **37.** $\log 9$ **38.** $\log 25$ **39.** $\log 45$
40. $\log 75$ **41.** $\log 27$ **42.** $\log \frac{1}{3}$ **43.** $\log \frac{1}{5}$
44. $\log 0.04$ **45.** $\log \frac{9}{10}$ **46.** $\log \frac{3}{5}$ **47.** $\log \left(\frac{3}{5}\right)^2$

In 48–55, if $\log a = c$, express each of the following in terms of c.

48. $\log a^2$ **49.** $\log 10a$ **50.** $\log 100\,a$ **51.** $\log \frac{a}{10}$
52. $\log \frac{100}{a}$ **53.** $\log \frac{a^2}{10}$ **54.** $\log \left(\frac{a}{10}\right)^2$ **55.** $\log \sqrt{a}$

56. Write the following expression as a single logarithm: $\log (x^2 - 4) + 2 \log 8 - \log 6$.

57. Write the following expression as a multiple, sum, and/or difference of logarithms: $\log \sqrt{\frac{xy}{z}}$.

Applying Skills

58. The formula $t = \frac{\log K}{0.045 \log e}$ gives the time t (in years) that it will take an investment P that is compounded continuously at a rate of 4.5% to increase to an amount K times the original principal.

a. Use the formula to complete the table to three decimal places.

K	1	2	3	4	5	10	20	30
t								

b. Use the table to graph the function $t = \frac{\log K}{0.045 \log e}$.

c. If Paul invests $1,000 in a savings account that is compounded continuously at a rate of 4.5%, when will his investment double? triple?

59. The pH (hydrogen potential) measures the acidity or alkalinity of a solution. In general, acids have pH values less than 7, while alkaline solutions (bases) have pH values greater than 7. Pure water is considered neutral with a pH of 7. The pH of a solution is given by the formula pH $= -\log x$ where x represents the hydronium ion concentration of the solution. Find, to the nearest hundredth, the approximate pH of each of the following:

a. Blood: $x = 3.98 \times 10^{-8}$

b. Vinegar: $x = 6.4 \times 10^{-3}$

c. A solution with $x = 4.0 \times 10^{-5}$

8-5 NATURAL LOGARITHMS

DEFINITION

A **natural logarithm** is a logarithm to the base e.

Recall that e is an irrational constant that is approximately equal to 2.718281828. If $x = e^y$, then $y = \log_e x$. The expression $\log_e x$ is written as **ln *x***. Therefore, $y = \ln x$ can be read as "y is the natural log of x" or as "y is the exponent to the base e of x" and is called the **natural logarithmic function**.

The TI-83+/84+ graphing calculator has a LN key that will return the natural logarithm of any number. For example, to find $\log_e 8$ or ln 8, enter the following sequence of keys.

ENTER: LN 8 ENTER

DISPLAY:

```
LN(8
          2.079441542
```

Therefore, $\ln 8 \approx 2.079441542$ or $e^{2.079441542} \approx 8$.

To show that this last statement is true, store the answer in your calculator and then use the e^x key, the 2nd function of the LN key.

ENTER: STO► X,T,Θ,*n* ENTER 2nd e^x X,T,Θ,*n* ENTER

DISPLAY:
```
Ans→X
          2.079441542
e^(X
                    8
```

EXAMPLE 1

Find ln 23.75.

Solution Use the LN key of the calculator.

ENTER: LN 23.75 ENTER

DISPLAY:
```
ln(23.75
           3.16758253
```

Answer 3.16758253

EXAMPLE 2

If $\ln x = 2.87534$, find x to the nearest tenth.

Solution Use the e^x key to find the antilogarithm.

ENTER: 2nd e^x 2.87534 ENTER

DISPLAY:
```
e^(2.87534
           17.73145179
```

Answer To the nearest tenth, $\ln 17.7 = 2.87534$ or $e^{2.87534} = 17.7$.

EXAMPLE 3

Use natural logs to find the value of $\frac{4.20^3 \times 0.781}{4.83}$ to the nearest tenth.

Solution

$$\ln \frac{4.20^3 \times 0.781}{4.83} = 3 \ln 4.20 + \ln 0.781 - \ln 4.83$$

ENTER: 3 LN 4.2) + LN .781) − LN 4.83 ENTER

DISPLAY:
```
3LN(4.2)+LN(.781
)-LN(4.83
          2.483226979
```

Now store this value for x and find its antilogarithm.

ENTER: STO▶ X,T,Θ,*n* ENTER 2nd e^x X,T,Θ,*n* ENTER

DISPLAY:
```
Ans→X
          2.483226979
e^(X
          11.97986087
```

Answer To the nearest tenth, $\frac{4.20^3 \times 0.781}{4.83} = 12.0$.

Exercises

Writing About Mathematics

1. Compare Example 3 in Section 8-4 with Example 3 in Section 8-5. Explain why the answers are the same.

2. For what value of a does $\log a = \ln a$? Justify your answer.

Developing Skills

In 3–14, find the natural logarithm of each number to the nearest hundredth.

3. 3.75 **4.** 8.56 **5.** 47.88 **6.** 56.2

7. 562 **8.** 5,620 **9.** 0.342 **10.** 0.0759

11. 1 **12.** e **13.** e^2 **14.** $\frac{1}{e}$

In 15–20, evaluate each logarithm to the nearest hundredth.

15. $\ln \frac{1}{2}$ **16.** $\ln 5 + \ln 7$ **17.** $\frac{1}{2} \ln 3 - \frac{1}{2} \ln 1$

18. $\ln 1{,}000^2$ **19.** $\frac{\ln 6 - \ln e}{2 \ln 8}$ **20.** $\frac{\ln \sqrt{5}}{\ln 10}$

In 21–32, for each given logarithm, find x, the antilogarithm. Write the answer to four decimal places.

21. $\ln x = 0.5787$ **22.** $\ln x = 0.8297$ **23.** $\ln x = 1.3826$

24. $\ln x = 1.7790$ **25.** $\ln x = 2.2030$ **26.** $\ln x = 2.5619$

27. $\ln x = 4.8200$ **28.** $\ln x = -0.5373$ **29.** $\ln x = -0.05729$

30. $\ln x = -1.1544$ **31.** $\ln x = -1$ **32.** $\ln x = -2$

In 33–44, if $\ln 2 = x$ and $\ln 3 = y$, write each of the natural logs in terms of x and y.

33. $\ln 6$ **34.** $\ln 9$ **35.** $\ln 4$ **36.** $\ln 12$

37. $\ln 24$ **38.** $\ln 36$ **39.** $\ln \frac{1}{3}$ **40.** $\ln \frac{1}{2}$

41. $\ln \frac{1}{6}$ **42.** $\ln \frac{1}{36}$ **43.** $\ln \frac{2}{3}$ **44.** $\ln \left(\frac{2}{3}\right)^2$

In 45–52, if $\ln a = c$, express each of the following in terms of c.

45. $\ln a^2$ **46.** $\ln a^3$ **47.** $\ln a^{-1}$ **48.** $\ln \frac{1}{a}$

49. $\ln a^{-2}$ **50.** $\ln \frac{1}{a^2}$ **51.** $\ln a^{\frac{1}{2}}$ **52.** $\ln \sqrt{a}$

In 53–56, find each value of x to the nearest thousandth.

53. $e^x = 35$ **54.** $e^x = 217$ **55.** $e^x = 2$ **56.** $e^x = -2$

57. Write the following expression as a single logarithm: $\frac{1}{2}\ln x - \ln y + \ln z^3$.

58. Write the following expression as a multiple, sum, and/or difference of logarithms: $\ln \frac{e^2 x y^{\frac{1}{2}}}{z}$.

Hands-On Activity: The Change of Base Formula

You may have noticed that your calculator can only evaluate common and natural logarithms. For logarithms with other bases, derive a formula by using the following steps:

1. Let $x = \log_b y$.
2. If $x = \log_b y$, then $b^x = y$ and $\log b^x = \log y$. Use the rule for the logarithm of a power to rewrite the left side of this last equation and solve for x.
3. Write $\log_b y$ by equating the value of x from step 2 and the value of x from step 1. This equation expresses the logarithm to the base b of y in terms of the common logarithm of y and the common logarithm of b.
4. Repeat steps 2 and 3 using $\ln b^x = \ln y$.

The equation in step 4 expresses the logarithm to the base b of y in terms of the natural logarithm of y and the natural logarithm of b.

These steps lead to the following equations:

$$\log_b y = \frac{\log y}{\log b} \qquad \log_b y = \frac{\ln y}{\ln b}$$

In **a**–**h**, use the calculator to evaluate the following logarithms to the nearest hundredth using both common and natural logarithms.

a. $\log_5 30$ **b.** $\log_2 17$ **c.** $\log_4 5$ **d.** $\log_{3.5} 10$

e. $\log_9 0.2$ **f.** $\log_{\frac{1}{2}} 6$ **g.** $\log_4 (12 \times 80)$ **h.** $\log_5 \frac{15}{28}$

8-6 EXPONENTIAL EQUATIONS

In chapter 7, we solved exponential equations by writing each side of the equation to the same base. Often that is possible only by using logarithms.

Since $f(x) = \log_b x$ is a function:

► **If $x_1 = x_2$, then $\log_b x_1 = \log_b x_2$.**

For example, solve $8^x = 32$ for x. There are two possible methods.

METHOD 1

Write each side of the equation to the base 2.

$$8^x = 32$$
$$(2^3)^x = 2^5$$
$$3x = 5$$
$$x = \frac{5}{3}$$

METHOD 2

Take the log of each side of the equation and solve for the variable.

$$8^x = 32$$
$$\log 8^x = \log 32$$
$$x \log 8 = \log 32$$
$$x = \frac{\log 32}{\log 8}$$

ENTER: LOG 32) ÷ LOG 8 ENTER MATH ENTER ENTER

DISPLAY:

```
LOG(32)/LOG(8
          1.666666667
Ans▶Frac
                  5/3
```

Check: $8^{\frac{5}{3}} = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32$ ✔

For exponential equations with different bases, we use logarithms. For example, solve $5^x = 32$ for x. In this equation, there is no base of which both 5 and 32 are a power.

$$5^x = 32$$
$$\log 5^x = \log 32$$
$$x \log 5 = \log 32$$
$$x = \frac{\log 32}{\log 5}$$

ENTER: LOG 32) ÷ LOG 5 ENTER

DISPLAY:
```
LOG(32)/LOG(5
        2.15338279
```

Check

ENTER: STO▶ X,T,Θ,n ENTER
5 ^ X,T,Θ,n ENTER

DISPLAY:
```
Ans→X
        2.15338279
5^X
                32
```

When solving an exponential equation for a variable, either common logs or natural logs can be used.

EXAMPLE 1

Solve for x to the nearest hundredth: $5.00(7.00)^x = 1{,}650$.

Solution *How to Proceed*

(1) Write the equation: $5.00(7.00)^x = 1{,}650$

(2) Write the natural log of each side of the equation: $\ln 5.00(7.00)^x = \ln 1{,}650$

(3) Simplify the equation:
$$\ln 5.00 + \ln 7.00^x = \ln 1{,}650$$
$$\ln 5.00 + x \ln 7.00 = \ln 1{,}650$$

(4) Solve the equation for x:
$$x \ln 7 = \ln 1{,}650 - \ln 5.00$$
$$x = \frac{\ln 1{,}650 - \ln 5.00}{\ln 7.00}$$

(5) Use a calculator to compute x:

ENTER: (LN 1650) − LN 5)) ÷ LN 7 ENTER

DISPLAY:
```
(LN(1650)-LN(5))
/LN(7
        2.980144102
```

Answer $x \approx 2.98$

EXAMPLE 2

If a \$100 investment receives 5% interest each year, after how many years will the investment have doubled in value?

Solution Let A dollars represent the value of an investment of P dollars after t years at interest rate r.

Since interest is compounded yearly, $A = P(1 + r)^t$.

If the value of the investment doubles, then $A = 2P$.

$P = 100$, $A = 2P = 200$, and $(1 + r) = (1 + 0.05) = (1.05)$.

Solve for t:

$$200 = 100(1.05)^t$$
$$2 = (1.05)^t$$
$$\log 2 = t \log 1.05$$
$$\frac{\log 2}{\log (1.05)} = t$$

ENTER: LOG 2) ÷ LOG 1.05) ENTER

DISPLAY:

```
LOG(2)/LOG(1.05)
         14.20669908
```

The investment will have almost doubled after 14 years and will have doubled in value in the 15th year.

Answer 15 years

EXAMPLE 3

The element fermium has a decay constant of −0.00866 days. After how many days will 7.0 grams remain of a 10.0-gram sample?

Solution Since an element decays constantly, use the equation $A_n = A_0e^{rt}$.

$A_0 = 10.0$, $A_n = 7.0$, $r = -0.00866$.

$$7.0 = 10.0\, e^{-0.00866t}$$
$$0.70 = e^{-0.00866t}$$
$$\ln 0.70 = \ln e^{-0.00866t}$$
$$\ln 0.70 = -0.00866t$$
$$\frac{\ln 0.70}{-0.00866} = t$$

ENTER: LN .7) ÷ (-) .00866 ENTER

DISPLAY:
```
ln(.7)/-.00866
       41.18648313
```

The sample will decay to 7.0 grams early on the 42nd day.

Answer 42 days

Exercises

Writing About Mathematics

1. Trisha said that the equation $5^x + 6 = 127$ could be solved by writing the logarithmic equation $x \log 5 + \log 6 = \log 127$. Do you agree with Trisha? Explain why or why not.

2. Melita said that the equation $4(3)^x = 72$ could be solved by writing the logarithmic equation $x \log 12 = \log 72$. Do you agree with Melita? Explain why or why not.

Developing Skills

In 3–14, solve each equation for the variable. Express each answer to the nearest hundredth.

3. $3^x = 12$
4. $2^b = 18$
5. $5^y = 100$
6. $10^x = 50$
7. $12^a = 254$
8. $6(3^x) = 532$
9. $7(2^b) = 815$
10. $5(10^y) = 1{,}200$
11. $(2 \times 8)^x = 0.25$
12. $(5 \times 7)^a = 0.585$
13. $12 + 9^x = 122$
14. $75 - 4^b = 20$

Applying Skills

15. When Rita was five, she had \$1 in her piggy bank. The next year she doubled the amount that she had in her piggy bank to \$2. She decided that each year she would double the amount in her piggy bank. How old will Rita be when she has at least \$1,000 in her piggy bank?

16. An investment of \$2,000 receives 5% interest annually. After how many years has the investment increased to at least \$2,500?

17. When interest is compounded quarterly (4 times a year) at an annual rate of 6%, the rate of interest for each quarter is $\frac{0.06}{4}$, and the number of times that interest is added in t years is $4t$. After how many years will an investment of \$100 compounded quarterly at 6% annually be worth at least \$450? (Use the formula $A_n = A_0\left(1 + \frac{r}{n}\right)^{nt}$.)

18. After how many years will \$100 invested at an annual rate of 6% compounded continuously be worth at least \$450? (Use the formula $A_n = A_0e^{rt}$.)

19. The decay constant of francium is −0.0315 minutes.

a. After how many minutes will 1.25 grams of francium remain of a 10.0-gram sample? Assume the exponential decay occurs continuously.

b. What is the half-life of francium? (The half-life of an element is the length of time needed for half of a sample to decay. For example, it is the length of time for a sample of 10 grams to be reduced to 5 grams of the original element.)

20. The half-life of einsteinium is 276 days.

a. To five decimal places, what is the decay constant of einsteinium? Assume the exponential decay occurs continuously.

b. After how many days will 2.5 grams of einsteinium remain of a sample of 20 grams?

8-7 LOGARITHMIC EQUATIONS

We have solved equations involving exponents by equating the log of each side of the equation. An equation given in terms of logs can be solved by equating the antilog of each side of the equation.

Since $y = \log_b x$ is a one-to-one function:

► **If $\log_b x_1 = \log_b x_2$, then $x_1 = x_2$.**

EXAMPLE 1

Solve for x and check: $\ln 12 - \ln x = \ln 3$.

Solution *How to Proceed*

(1) Write the equation: $\ln 12 - \ln x = \ln 3$

(2) Solve for $\ln x$: $-\ln x = -\ln 12 + \ln 3$

$\ln x = \ln 12 - \ln 3$

(3) Simplify the right side of the equation: $\ln x = \ln \frac{12}{3}$

(4) Equate the antilog of each side of the equation: $x = \frac{12}{3}$

$x = 4$

Alternative Solution *How to Proceed*

(1) Write the equation: $\ln 12 - \ln x = \ln 3$

(2) Simplify the left side of the equation: $\ln \frac{12}{x} = \ln 3$

(3) Equate the antilog of each side of the equation: $\frac{12}{x} = 3$

(4) Solve for x: $12 = 3x$

$4 = x$

Check

$$\ln 12 - \ln x = \ln 3$$
$$\ln 12 - \ln 4 \stackrel{?}{=} \ln 3$$
$$\ln \frac{12}{4} \stackrel{?}{=} \ln 3$$
$$\ln 3 = \ln 3 \checkmark$$

Calculator check

ENTER: LN 12) − LN 4 ENTER

LN 3 ENTER

DISPLAY:

```
LN(12)-LN(4
          1.098612289
LN(3
          1.098612289
```

Answer $x = 4$

Note that in both methods of solution, each side must be written as a single log before taking the antilog of each side.

EXAMPLE 2

Solve for x: $\log x + \log (x + 5) = \log 6$.

Solution

(1) Write the equation: $\log x + \log (x + 5) = \log 6$

(2) Simplify the left side: $\log [x(x + 5)] = \log 6$

(3) Equate the antilog of each side of the equation: $x(x + 5) = 6$

(4) Solve the equation for x:

$$x^2 + 5x = 6$$
$$x^2 + 5x - 6 = 0$$
$$(x - 1)(x + 6) = 0$$

$$x - 1 = 0 \quad\Big|\quad x + 6 = 0$$
$$x = 1 \quad\Big|\quad x = -6 \;✗$$

Reject the negative root. In the given equation, $\log x$ is only defined for positive values of x.

Check

$$\log x + \log (x + 5) = \log 6$$
$$\log 1 + \log (1 + 5) \stackrel{?}{=} \log 6$$
$$0 + \log 6 \stackrel{?}{=} \log 6$$
$$\log 6 = \log 6 \checkmark$$

Answer $x = 1$

EXAMPLE 3

Solve for b: $\log_b 8 = \log_4 64$.

Solution Let each side of the equation equal x.

Let $x = \log_4 64$.	Let $x = \log_b 8$.
$4^x = 64$	$3 = \log_b 8$
$(2^2)^x = 2^6$	$b^3 = 8$
$2^{2x} = 2^6$	$(b^3)^{\frac{1}{3}} = 8^{\frac{1}{3}}$
$2x = 6$	$b = \sqrt[3]{8}$
$x = 3$	$b = 2$

Check

$$\log_b 8 = \log_4 64$$
$$\log_2 8 \stackrel{?}{=} \log_4 64$$
$$3 = 3 \checkmark$$

Answer $b = 2$

Exercises

Writing About Mathematics

1. Randall said that the equation $\log x + \log 12 = \log 9$ can be solved by writing the equation $x + 12 = 9$. Do you agree with Randall? Explain why or why not.

2. Pritha said that before an equation such as $\log x = 1 + \log 5$ can be solved, 1 could be written as $\log 10$. Do you agree with Pritha? Explain why or why not.

Developing Skills

In 3–14, solve each equation for the variable and check.

3. $\log x + \log 8 = \log 200$

4. $\log x + \log 15 = \log 90$

5. $\ln x + \ln 18 = \ln 27$

6. $\log x - \log 5 = \log 6$

7. $\log x - \log 3 = \log 42$

8. $\ln x - \ln 24 = \ln 8$

9. $\log 8 - \log x = \log 2$

10. $\log (x + 3) = \log (x - 5) + \log 3$

11. $\log x + \log (x + 7) = \log 30$

12. $\log x + \log (x - 1) = \log 12$

13. $2 \log x = \log 25$

14. $3 \ln x + \ln 24 = \ln 3$

In 15–18, find x to the nearest hundredth.

15. $\log x - 2 = \log 5$

16. $\log x + \log (x + 2) = \log 3$

17. $2 \log x = \log (x - 1) + \log 5$

18. $2 \log x = \log (x + 3) + \log 2$

CHAPTER SUMMARY

A **logarithm** is an exponent: $x = b^y \leftrightarrow y = \log_b x$. The expression $y = \log_b x$ can be read as "y is the logarithm to the base b of x."

For $b > 0$ and $b \neq 1$, $f(x) = b^x$, then $f^{-1}(x) = \log_b x$, the **logarithmic function**. The domain of $f^{-1}(x) = \log_b x$ is the set of positive real numbers and the range is the set of real numbers. The y-axis is the **vertical asymptote** of $f^{-1}(x) = \log_b x$.

Because a logarithm is an exponent, the rules for logarithms are derived from the rules for exponents. If $b^x = c$ and $b^y = d$:

	Powers	Logarithms
Exponent of Zero	$b^0 = 1$	$\log_b 1 = 0$
Exponent of One	$b^1 = b$	$\log_b b = 1$
Products	$b^{x+y} = cd$	$\log_b cd = \log_b c + \log_b d$ $= x + y$
Quotients	$b^{x-y} = \frac{c}{d}$	$\log_b \frac{c}{d} = \log_b c - \log_b d$ $= x - y$
Powers	$(b^x)^a = b^{ax} = c^a$	$\log_b c^a = a \log_b c$ $= ax$

A **common logarithm** is a logarithm to the base 10. When no base is written, the logarithm is a common logarithm: $\log A = \log_{10} A$.

For $\log_b x = y$, x is the **antilogarithm** of y.

A **natural logarithm** is a logarithm to the base e. A natural logarithm is abbreviated ln: $\ln A = \log_e A$.

The rules for logarithms are used to simplify exponential and logarithmic equations:

- If $\log_b x_1 = \log_b x_2$, then $x_1 = x_2$.
- If $x_1 = x_2$, then $\log_b x_1 = \log_b x_2$.

VOCABULARY

8-1 Logarithm • Logarithm function • Vertical asymptote

8-4 Common logarithm • Antilogarithm

8-5 Natural logarithm • ln x • Natural logarithmic function

REVIEW EXERCISES

1. a. Sketch the graph of $f(x) = \log_3 x$.

b. What is the domain of $f(x)$?

c. What is the range of $f(x)$?

d. Sketch the graph of $f^{-1}(x)$.

e. Write an equation for $f^{-1}(x)$.

In 2–4, solve each equation for y in terms of x.

2. $x = \log_6 y$ **3.** $x = \log_{2.5} y$ **4.** $x = 82^y$

In 5–10, write each expression in logarithmic form.

5. $2^3 = 8$ **6.** $6^2 = 36$ **7.** $10^{-1} = 0.1$

8. $3^{\frac{1}{2}} = \sqrt{3}$ **9.** $4 = 8^{\frac{2}{3}}$ **10.** $\frac{1}{4} = 2^{-2}$

In 11–16, write each expression in exponential form.

11. $\log_3 81 = 4$ **12.** $\log_5 125 = 3$ **13.** $\log_4 8 = \frac{3}{2}$

14. $\log_7 \sqrt{7} = \frac{1}{2}$ **15.** $-1 = \log 0.1$ **16.** $0 = \ln 1$

In 17–20, evaluate each logarithmic expression. Show all work.

17. $3 \log_2 8$ **18.** $\frac{14}{9} \log_5 625$

19. $\log_3 \frac{1}{27} \div \log_9 \sqrt{3}$ **20.** $\dfrac{\log_2 256 \cdot \log_{861} 861^{\frac{1}{4}}}{4 \log_{13} \frac{1}{169}}$

21. If $f(x) = \log_3 x$, find $f(27)$. **22.** If $f(x) = \log x$, find $f(0.01)$.

23. If $f(x) = \log_4 x$, find $f(32)$. **24.** If $f(x) = \ln x$, find $f(e^4)$.

In 25–32, if $\log 5 = a$ and $\log 3 = b$, express each log in terms of a and b.

25. $\log 15$ **26.** $\log 25$ **27.** $\log 5(3)^2$ **28.** $\log \sqrt{45}$

29. $\log \frac{\sqrt{3}}{5}$ **30.** $\log \left(\frac{3}{5}\right)^2$ **31.** $\log \sqrt[3]{\frac{5}{3}}$ **32.** $\log \frac{25 \times \sqrt{3}}{9}$

33. If $b > 1$, then what is the value of $\log_b b$?

34. If $\log a = 0.5$, what is $\log (100a)$?

In 35–40, solve each equation for the variable. Show all work.

35. $x = \log_4 32$ **36.** $\log_b 27 = \frac{3}{2}$ **37.** $\log_6 x = -2$

38. $\log 0.001 = x$ **39.** $\log_{25} x = \frac{1}{4}$ **40.** $\log_b 16 = -2$

In 41–44: **a.** Write each expression as a single logarithm. **b.** Find the value of each expression.

41. $\frac{1}{4}\log_4 81 - \log_4 48$ **42.** $\log_{360} 5 + \log_{360} 12 + \log_{360} 6$

43. $\log_{0.5} 64 + \log_{0.5} 0.25 - 8\log_{0.5} 2$ **44.** $\log_{1.5} \frac{3}{2} + \log_{1.5} 3 + \log_{1.5} \frac{1}{2}$

In 45–47: **a.** Expand each expression using the properties of logarithms. **b.** Evaluate each expression to the nearest hundredth.

45. $\ln \frac{42^2}{3}$ **46.** $\ln (14^2 \cdot 0.625)$ **47.** $\ln (0.25^4 \div (26 \cdot 3^{-5}))$

In 48–53, write an equation for A in terms of x and y.

48. $\log A = \log x + \log y$ **49.** $\log_2 A = \log_2 x - \log_2 y$

50. $\ln x = \ln y - \ln A$ **51.** $\log_5 A = \log_5 x + 3\log_5 y$

52. $\log A = 2(\log x - \log y)$ **53.** $\ln x = \ln A - \frac{1}{3}\ln y$

In 54–56, if $\log x = 1.5$, find the value of each logarithm.

54. $\log 100x$ **55.** $\log x^2$ **56.** $\log \sqrt{x}$

57. If $\log x = \log 3 - \log (x - 2)$, find the value of x.

58. For what value of a does $1 + \log_3 a - \log_3 2 = \log_3 (a + 1)$?

59. If $2 \log (x + 1) = \log 5$, find x to the nearest hundredth.

60. In 2000, it was estimated that there were 50 rabbits in the local park and that the number of rabbits was growing at a rate of 8% per year. If the estimates are correct and the rabbit population continues to increase at the estimated rate, in what year will there be three times the number of rabbits, that is, 150 rabbits? (If A_0 = the number of rabbits present in 2000, A_n = the number of rabbits n years later and k = the annual rate of increase, $A_n = A_0e^{kn}$.)

61. When Tobey was born, his parents invested \$2,000 in a fund that paid an annual interest of 6%. How old will Tobey be when the investment is worth at least \$5,000?

62. The population of a small town is decreasing at the rate of 2% per year. The town historian records the population at the end of each year. In 2000 ($n = 0$), the population was 5,300. If this decrease continued, what would have been the population, to the nearest hundred, in 2008 ($n = 8$)? (Use $P_n = P_0(1 + r)^n$ where P_0 is the population when $n = 0$, P_n is the population in n years, and r is the rate of change.)

Exploration

Using tables of logarithms was the common method of calculation before computers and calculators became everyday tools. These tables were relatively simple to use but their development was a complex process. John Napier spent many years developing these tables, often referred to as *Napier's bones*. Napier observed a relationship between arithmetic and geometric sequences. To distinguish the arithmetic sequence from the geometric sequence, we will let a_n be the terms of the arithmetic sequence and g_n be the terms of the geometric sequence.

For example, compare the arithmetic sequence with $a_1 = 0$ and $d = 1$ with the geometric sequence with $g_1 = 1$ and $r = 10$.

a_n	0	1	2	3	4	5	...
g_n	1	10	100	1,000	10,000	100,000	...

Note that for all n, $g_n = 10^{a_n}$ or $\log g_n = a_n$.

The arithmetic mean between a_1 and a_2, that is, between 0 and 1, is $\frac{0 + 1}{2} = 0.5$. The geometric mean between g_1 and g_2 is the mean proportional between the first and second terms of the sequence:

$$\frac{1}{x} = \frac{x}{10} \rightarrow x^2 = (1)(100) \text{ or } x = \sqrt{(1)(100)} \approx 3.16227766$$

For these two means, $3.16227766 = 10^{0.5}$ or $\log 3.16227766 = 0.5$.

Now use a recursive method. Use the results of the last step to find a new log. The arithmetic mean of 0 and 0.5 is $\frac{0 + 0.5}{2} = 0.25$. The geometric mean between 1 and 3.16227766 is 1.77827941. For these two means, $1.77827941 = 10^{0.25}$ or $\log 1.77827941 = 0.25$.

Complete the following steps. Use a calculator to find the geometric means.

a_n	g_n
0	1
1	10
(7)	(8)
(4)	(5)
(1)	(2)
2	100

STEP 1. Find the arithmetic mean of $a_2 = 1$ and $a_3 = 2$ and place it in slot (1) in the table on the left.

STEP 2. Find the geometric mean of $g_2 = 10$ and $g_3 = 100$ and place it in slot (2) in the table on the left.

STEP 3. Show that the log of the geometric mean found in step 2 is equal to the arithmetic mean found in step 1.

STEP 4. Repeat step 1 using $a_2 = 1$ and the mean in slot (1) and place it in slot (4).

STEP 5. Repeat step 2 using $g_2 = 10$ and the mean in slot (2) and place it in slot (5).

STEP 6. Show that the log of the geometric mean found in step 5 is equal to the arithmetic mean found in step 4.

STEP 7. Repeat step 1 using $a_2 = 1$ and the mean in slot (4) and place it in slot (7).

STEP 8. Repeat step 2 using $g_2 = 10$ and the mean in slot (5) and place it in slot (8).

STEP 9. Show that the log of the geometric mean found in step 8 is equal to the arithmetic mean found in step 7.

STEP 10. The geometric mean of 17.7827941 and 31.6227766 is 23.71373706. Find log 23.71373706 without using the **LOG** or **LN** keys of the calculator.

CUMULATIVE REVIEW — CHAPTERS 1–8

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If the range of f(x) is a subset of the set of real numbers, which of the following is not an element of the domain of $f(x) = \sqrt{9 - x^2}$?

(1) 0 (2) 1 (3) 3 (4) 4

2. An equation with imaginary roots is

(1) $x^2 - 2x - 5 = 0$
(2) $x^2 - 2x + 5 = 0$
(3) $x^2 + 2x - 5 = 0$
(4) $x^2 + 2x = 5$

3. The solution set of $5 - \sqrt{x + 1} = x$ is

(1) {3, 8} (2) {−8, −3} (3) {8} (4) {3}

4. If $f(x) = 2x + 3$ and $g(x) = x^2$, then $f(g(-2))$ is equal to

(1) −4 (2) 1 (3) 4 (4) 11

5. Which of the following is an arithmetic sequence?

(1) 1, 3, 9, 27, 81
(2) 1, 2, 4, 7, 11
(3) 1, 3, 6, 12, 24
(4) 1, 4, 7, 10, 13

6. The expression $(3 - i)^2$ is equivalent to

(1) 8 (2) $8 - 6i$ (3) 10 (4) $8 + 6i$

7. The 9th term of the sequence 13, 9, 5, 1, . . . is

(1) −36 (2) −32 (3) −23 (4) −19

8. The fraction $\frac{2 + \sqrt{3}}{2 - \sqrt{3}}$ is equivalent to

(1) $11\sqrt{3}$ (2) $7 - 4\sqrt{3}$ (3) $7 + 4\sqrt{3}$ (4) $\frac{7 + 4\sqrt{3}}{7}$

9. The sum $2(3)^{-2} + 6^{-1}$ is equal to

(1) $\frac{7}{36}$ (2) $\frac{7}{18}$ (3) $\frac{1}{6^3}$ (4) $2(3)^{-3}$

10. $\sum_{i=0}^{3} 2^i i$ is equal to

(1) 24 (2) 26 (3) 34 (4) 90

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Find all the zeros of the function $f(x) = 4x^3 - x$.

12. The sum of the roots of a quadratic equation is 10 and the product is 34. Write the equation and find the roots.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Find, to the nearest hundredth, the value of x such that $5^{3x} = 1{,}000$.

14. Express the sum of $\sqrt{200} + \sqrt{50} + 2\sqrt{8}$ in simplest radical form.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. a. Write $\sum_{n=1}^{8} 30(0.2)^{n-1}$ as the sum of terms.

b. Find $\sum_{n=1}^{8} 30(0.2)^{n-1}$ using a formula for the sum of a series.

16. Carbon-14 has a decay constant of -0.000124 when time is in years. A sample of wood that is estimated to have originally contained 15 grams of carbon-14 now contains 9.25 grams. Estimate, to the nearest hundred years, the age of the wood.