

SEQUENCES AND SERIES

When the Grant family purchased a computer for \$1,200 on an installment plan, they agreed to pay \$100 each month until the cost of the computer plus interest had been paid. The interest each month was 1.5% of the unpaid balance. The amount that the Gant family still owed after each payment is a function of the number of months that have passed since they purchased the computer.

- At the end of month 1:
they owed $(1,200 + 1,200 \times 0.015 - 100)$ dollars or
 $(1,200 \times 1.015 - 100) = \$1,118$.
- At the end of month 2:
they owed $(1,118 + 1,118 \times 0.015 - 100)$ dollars or
 $(1,118 \times 1.015 - 100) = \$1,034.77$.

Each month, interest is added to the balance from the previous month and a payment of \$100 is subtracted.

We can express the monthly payments with the function defined as $\{(n, f(n))\}$. Let the domain be the set of positive integers that represent the number of months since the initial purchase. Then:

$$\begin{aligned} f(1) &= 1118 \\ f(2) &= 1034.77 \end{aligned}$$

In general, for positive integers n :

$$f(n) = f(n - 1) \times 1.015 - 100$$

This pattern continues until $(f(n - 1) \times 1.015)$ is between 0 and 100, since the final payment would be $(f(n - 1) \times 1.015)$ dollars.

In this chapter we will study sequential functions, such as the one described here, whose domain is the set of positive integers.

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6-1 SEQUENCES

A ball is dropped from height of 16 feet. Each time that it bounces, it reaches a height that is half of its previous height. We can list the height to which the ball bounces in order until it finally comes to rest.

After Bounce	1	2	3	4	5	6	7
Height (ft)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The numbers 8, 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$ form a *sequence*. A **sequence** is a set of numbers written in a given order. We can list these heights as ordered pairs of numbers in which each height is paired with the number that indicates its position in the list. The set of ordered pairs would be:

$$\{(1, 8), (2, 4), (3, 2), (4, 1), (5, 0.5), (6, 0.25), (7, 0.125)\}$$

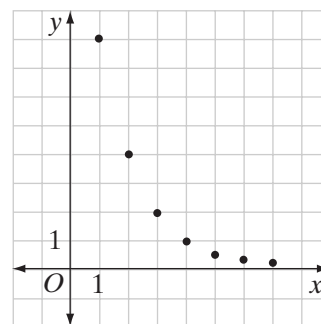
We associate each term of the sequence with the positive integer that specifies its position in the ordered set. Therefore, a sequence is a special type of function.

DEFINITION

A **finite sequence** is a function whose domain is the set of integers $\{1, 2, 3, \dots, n\}$.

The function that lists the height of the ball after 7 bounces is shown on the graph at the right.

Often the sequence can continue without end. In this case, the domain is the set of positive integers.

**DEFINITION**

An **infinite sequence** is a function whose domain is the set of positive integers.

The terms of a sequence are often designated as $a_1, a_2, a_3, a_4, a_5, \dots$. If the sequence is designated as the function f , then $f(1) = a_1, f(2) = a_2$, or in general:

$$f(n) = a_n$$

Most sequences are sets of numbers that are related by some pattern that can be expressed as a formula. The formula that allows any term of a sequence, except the first, to be computed from the previous term is called a **recursive definition**.

For example, the sequence that lists the heights to which a ball bounces when dropped from a height of 16 feet is 8, 4, 2, 1, 0.5, 0.25, 0.125, In this sequence, each term after the first is $\frac{1}{2}$ the previous term. Therefore, for each term after the first,

$$4 = \frac{1}{2}(8), 2 = \frac{1}{2}(4), 1 = \frac{1}{2}(2), 0.5 = \frac{1}{2}(1), 0.25 = \frac{1}{2}(0.5), 0.125 = \frac{1}{2}(0.25), \dots$$

For $n > 1$, we can write the recursive definition:

$$a_n = \frac{1}{2}a_{n-1}$$

Alternatively, we can write the recursive definition as:

$$a_{n+1} = \frac{1}{2}a_n \text{ for } n \geq 1$$

A rule that designates any term of a sequence can often be determined from the first few terms of the sequence.

EXAMPLE 1

- List the next three terms of the sequence 2, 4, 8, 16,
- Write a general expression for a_n .
- Write a recursive definition for the sequence.

Solution

- It appears that each term of the sequence is a power of 2: $2^1, 2^2, 2^3, 2^4, \dots$. Therefore, the next three terms should be $2^5, 2^6$, and 2^7 or 32, 64, and 128.
- Each term is a power of 2 with the exponent equal to the number of the term. Therefore, $a_n = 2^n$.
- Each term is twice the previous term. Therefore, for $n > 1$, $a_n = 2a_{n-1}$. Alternatively, for $n \geq 1$, $a_{n+1} = 2a_n$.

Answers a. 32, 64, 128

b. $a_n = 2^n$

c. For $n > 1$, $a_n = 2a_{n-1}$ or for $n \geq 1$, $a_{n+1} = 2a_n$. ■

EXAMPLE 2

Write the first three terms of the sequence $a_n = 3n - 1$.

Solution $a_1 = 3(1) - 1 = 2$ $a_2 = 3(2) - 1 = 5$ $a_3 = 3(3) - 1 = 8$

Answer The first three terms of the sequence are 2, 5, and 8. ■

Sequences and the Graphing Calculator



We can use the sequence function on the graphing calculator to view a sequence for a specific range of terms. Evaluate a *sequence* in terms of *variable* from a *beginning term* to an *ending term*. That is:

$$\text{seq}(\text{sequence}, \text{variable}, \text{beginning term}, \text{ending term})$$

For example, we can examine the sequence $a_n = n^2 + 2$ on the calculator. To view the first 20 terms of the sequence:

ENTER: 2nd LIST ► 5 X,T,θ,n x² + 2 ,
X,T,θ,n , 1 , 20) ENTER

DISPLAY: seq(X²+2,X,1,20)
{3 6 11 18 27 3...

Note: We use the variable X instead of n to enter the sequence. We can also use the left and right arrow keys to examine the terms of the sequence.

Exercises

Writing About Mathematics

1. Nichelle said that sequence of numbers in which each term equals half of the previous term is a finite sequence. Randi said that it is an infinite sequence. Who is correct? Justify your answer.
2. a. Jacob said that if $a_n = 3n - 1$, then $a_{n+1} = a_n + 3$. Do you agree with Jacob? Explain why or why not.
 b. Carlos said that if $a_n = 2^n$, then $a_{n+1} = 2^{n+1}$. Do you agree with Carlos? Explain why or why not.

Developing Skills

In 3–18, write the first five terms of each sequence.

- | | | | |
|------------------------|---------------------|---------------------------|-----------------------------|
| 3. $a_n = n$ | 4. $a_n = n + 5$ | 5. $a_n = 2n$ | 6. $a_n = \frac{1}{n}$ |
| 7. $a_n = \frac{n}{2}$ | 8. $a_n = 20 - n$ | 9. $a_n = 3^n$ | 10. $a_n = n^2$ |
| 11. $a_n = 2n + 3$ | 12. $a_n = 2n - 1$ | 13. $a_n = \frac{n}{n+1}$ | 14. $a_n = \frac{n+2}{n}$ |
| 15. $a_n = -n$ | 16. $a_n = 12 - 3n$ | 17. $a_n = \frac{4n}{3}$ | 18. $a_n = \frac{n}{2} + i$ |

In 19–30: **a.** Write an algebraic expression that represents a_n for each sequence. **b.** Find the ninth term of each sequence.

19. 2, 4, 6, 8, ...

20. 3, 6, 9, 12, ...

21. 1, 4, 7, 10, ...

22. 3, 9, 27, 81, ...

23. 12, 6, 3, 1.5, ...

24. 7, 9, 11, 13, ...

25. $10i, 8i, 6i, 4i, \dots$

26. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$

27. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots$

28. 2, 5, 10, 17, ...

29. 1, -2, 3, -4, ...

30. $1, \sqrt{2}, \sqrt{3}, 2, \dots$

In 31–39, write the first five terms of each sequence.

31. $a_1 = 5, a_n = a_{n-1} + 1$

32. $a_1 = 1, a_{n+1} = 3a_n$

33. $a_1 = 1, a_n = 2a_{n-1} + 1$

34. $a_1 = -2, a_n = -2a_{n-1}$

35. $a_1 = 20, a_n = a_{n-1} - 4$

36. $a_1 = 4, a_{n+1} = a_n + n$

37. $a_2 = 36, a_n = \frac{1}{3}a_{n-1}$

38. $a_3 = 25, a_{n+1} = 2.5a_n$

39. $a_5 = \frac{1}{2}, a_n = \frac{1}{a_{n-1}}$

Applying Skills

40. Sean has started an exercise program. The first day he worked out for 30 minutes. Each day for the next six days, he increased his time by 5 minutes.

a. Write the sequence for the number of minutes that Sean worked out for each of the seven days.

b. Write a recursive definition for this sequence.

41. Sherri wants to increase her vocabulary. On Monday she learned the meanings of four new words. Each other day that week, she increased the number of new words that she learned by two.

a. Write the sequence for the number of new words that Sherri learned each day for a week.

b. Write a recursive definition for this sequence.

42. Julie is trying to lose weight. She now weighs 180 pounds. Every week for eight weeks, she was able to lose 2 pounds.

a. List Julie's weight for each week.

b. Write a recursive definition for this sequence.

43. January 1, 2008, was a Tuesday.

a. List the dates for each Tuesday in January of that year.

b. Write a recursive definition for this sequence.

44. Hui started a new job with a weekly salary of \$400. After one year, and for each year that followed, his salary was increased by 10%. Hui left this job after six years.

a. List the weekly salary that Hui earned each year.

b. Write a recursive definition for this sequence.

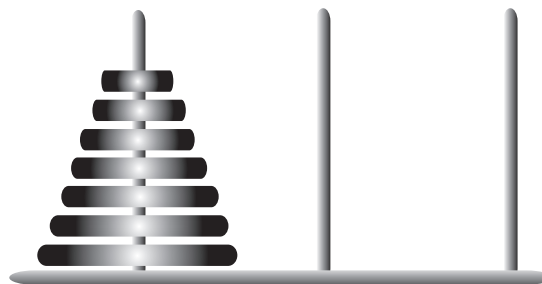
45. One of the most famous sequences is the Fibonacci sequence. In this sequence, $a_1 = 1, a_2 = 1$, and for $n > 2$, $a_n = a_{n-2} + a_{n-1}$. Write the first ten terms of this sequence.

Hands-On Activity

The Tower of Hanoi is a famous problem that has challenged problem solvers throughout the ages. The tower consists of three pegs. On one peg there are a number of disks of different sizes, stacked according to size with the largest at the bottom. The task is to move the entire stack from one peg to the other side using the following rules:

- Only one disk may be moved at a time.
- No disk may be placed on top of a smaller disk.

Note that a move consists of taking the top disk from one peg and placing it on another peg.



1. Use a stack of different-sized coins to model the Tower of Hanoi. What is the smallest number of moves needed if there are:
 - a. 2 disks?
 - b. 3 disks?
 - c. 4 disks?
 - d. n disks?
2. Write a recursive definition for the sequence described in Exercise 1, **a–d**.

6-2 ARITHMETIC SEQUENCES

The set of positive odd numbers, $1, 3, 5, 7, \dots$, is a sequence. The first term, a_1 , is 1 and each term is 2 greater than the preceding term. The difference between consecutive terms is 2. We say that 2 is the **common difference** for the sequence. The set of positive odd numbers is an example an *arithmetic sequence*.

DEFINITION

An **arithmetic sequence** is a sequence such that for all n , there is a constant d such that $a_n - a_{n-1} = d$.

For an arithmetic sequence, the recursive formula is:

$$a_n = a_{n-1} + d$$

An arithmetic sequence is formed when each term after the first is obtained by adding the same constant to the previous term. For example, look at the first five terms of the sequence of positive odd numbers.

$$\begin{array}{ll}
a_1 = 1 & a_1 = 1 \\
a_2 = a_1 + 2 = 1 + 2 = 3 & a_2 = 1 + 1(2) \\
a_3 = a_2 + 2 = 3 + 2 = 5 & a_3 = [1 + 1(2)] + 2 = 1 + 2(2) \\
a_4 = a_3 + 2 = 5 + 2 = 7 & a_4 = [1 + 2(2)] + 2 = 1 + 3(2) \\
a_5 = a_4 + 2 = 7 + 2 = 9 & a_5 = [1 + 3(2)] + 2 = 1 + 4(2)
\end{array}$$

For this sequence, $a_1 = 1$ and each term after the first is found by adding 2 to the preceding term. Therefore, for each term, 2 has been added to the first term one less time than the number of the term. For the second term, 2 has been added once; for the third term, 2 has been added twice; for the fourth term, 2 has been added three times. In general, for the n th term, 2 has been added $n - 1$ times.

Shown below is a general arithmetic sequence with a_1 as the first term and d as the common difference:

$$a_1, a_1 + d, a_1 + 2d, a_1 + 3d, a_1 + 4d, a_1 + 5d, \dots, a_1 + (n - 1)d$$

If the first term of an arithmetic sequence is a_1 and the common difference is d , then for each term of the sequence, d has been added to a_1 one less time than the number of the term. Therefore, any term a_n of an arithmetic sequence can be evaluated with the formula

$$a_n = a_1 + (n - 1)d$$

where d is the common difference of the sequence.

EXAMPLE I

For the arithmetic sequence 100, 97, 94, 91, \dots , find:

- the common difference.
- the 20th term of the sequence.

Solution **a.** The common difference is the difference between any term and the previous term:

$$97 - 100 = -3 \quad \text{or} \quad 94 - 97 = -3 \quad \text{or} \quad 91 - 94 = -3$$

The common difference is -3 .

$$\begin{aligned}
\text{b. } a_{20} &= 100 + (20 - 1)(-3) \\
&= 100 + (19)(-3) \\
&= 100 - 57 \\
&= 43
\end{aligned}$$

The 20th term can also be found by writing the sequence:

$$100, 97, 94, 91, 88, 85, 82, 79, 76, 73, 70, 67, 64, 61, 58, 55, 52, 49, 46, 43$$

EXAMPLE 2

Scott is saving to buy a guitar. In the first week, he put aside \$42 that he received for his birthday, and in each of the following weeks, he added \$8 to his savings. He needs \$400 for the guitar that he wants. In which week will he have enough money for the guitar?

Solution Let $a_1 = 42$ and $d = 8$.

The value for n for which $a_n = 400$ is the week Scott will have enough money to buy the guitar.

$$\begin{aligned}a_n &= a_1 + (n - 1)d \\400 &= 42 + (n - 1)(8) \\400 &= 42 + 8n - 8 \\400 &= 34 + 8n \\366 &= 8n \\45.75 &= n\end{aligned}$$

Scott adds money to his savings in \$8 increments, so he will not have the needed \$400 in savings until the 46th week.

Answer 46th week

EXAMPLE 3

The 4th term of an arithmetic sequence is 80 and the 12th term is 32.

- What is the common difference?
- What is the first term of the sequence?

Solution **a.** *How to Proceed*

$$\begin{array}{ll}(1) \text{ Use } a_n = a_1 + (n - 1)d \text{ to} & 80 = a_1 + (4 - 1)d \rightarrow 80 = a_1 + 3d \\ \text{write two equations in two} & 32 = a_1 + (12 - 1)d \rightarrow \underline{-(32 = a_1 + 11d)} \\ \text{variables:} & \\(2) \text{ Subtract to eliminate } a_1: & 48 = -8d \\(3) \text{ Solve for } d: & -6 = d\end{array}$$

b. Substitute in either equation to find a_1 .

$$\begin{aligned}80 &= a_1 + 3(-6) \\80 &= a_1 - 18 \\98 &= a_1\end{aligned}$$

Answers **a.** $d = -6$ **b.** $a_1 = 98$

Arithmetic Means

We have defined the *arithmetic mean* of two numbers as their average, that is, the sum of the numbers divided by 2. For example, the arithmetic mean of 4 and 28 is $\frac{4+28}{2} = \frac{32}{2} = 16$. The numbers 4, 16, and 28 form an arithmetic sequence with a common difference of 12.

We can find three numbers between 4 and 28 that together with 4 and 28 form an arithmetic sequence: $4, a_2, a_3, a_4, 28$. This is a sequence of five terms.

1. Use the formula for a_5 to find the common difference:

$$\begin{aligned}a_5 &= a_1 + (n - 1)d \\28 &= 4 + (5 - 1)d \\28 &= 4 + 4d \\24 &= 4d \\6 &= d\end{aligned}$$

2. Now use the recursive formula, $a_n = a_{n-1} + d$, and $d = 6$, to write the sequence:

$$4, 10, 16, 22, 28$$

The numbers 10, 16, and 22 are three **arithmetic means** between 4 and 28. Looking at it another way, in *any* arithmetic sequence, any given term is the average of the term before it and the term after it. Thus, an *arithmetic mean* is any term of an arithmetic sequence.

EXAMPLE 4

Find five arithmetic means between 2 and 23.

Solution Five arithmetic means between 2 and 23 will form a sequence of seven terms. Use the formula for a_7 to find the common difference:

$$\begin{aligned}a_7 &= a_1 + (n - 1)d \\23 &= 2 + (7 - 1)d \\23 &= 2 + 6d \\21 &= 6d \\\frac{7}{2} &= d\end{aligned}$$

Evaluate a_2, a_3, a_4, a_5 , and a_6 :

$$\begin{aligned}a_2 &= 2 + (2 - 1)\frac{7}{2} & a_3 &= 2 + (3 - 1)\frac{7}{2} & a_4 &= 2 + (4 - 1)\frac{7}{2} \\a_2 &= \frac{11}{2} = 5\frac{1}{2} & a_3 &= 9 & a_4 &= \frac{25}{2} = 12\frac{1}{2} \\a_5 &= 2 + (5 - 1)\frac{7}{2} & a_6 &= 2 + (6 - 1)\frac{7}{2} \\a_5 &= 16 & a_6 &= \frac{39}{2} = 19\frac{1}{2}\end{aligned}$$

Answer The five arithmetic means are $5\frac{1}{2}, 9, 12\frac{1}{2}, 16$, and $19\frac{1}{2}$. 

Exercises

Writing About Mathematics

1. Virginia said that Example 3 could have been solved without using equations. Since there are eight terms from a_4 to a_{12} , the difference between 80 and 32 has to be divided into eight parts. Each part is 6. Since the sequence is decreasing, the common difference is -6 . Then using -6 , work back from a_4 to a_1 : 80, 86, 92, 98. Do you think that Virginia's solution is better than the one given in Example 3? Explain why or why not.
2. Pedro said that to form a sequence of five terms that begins with 2 and ends with 12, you should divide the difference between 12 and 2 by 5 to find the common difference. Do you agree with Pedro? Explain why or why not.

Developing Skills

In 3–8, determine if each sequence is an arithmetic sequence. If the sequence is arithmetic, find the common difference.

3. 2, 5, 8, 11, 14, ...

4. $-3i, -1i, 1i, 3i, 5i, \dots$

5. 1, 1, 2, 3, 5, 8, ...

6. 20, 15, 10, 5, 0, ...

7. 1, 2, 4, 8, 16, ...

8. 1, 1.25, 1.5, 1.75, 2, ...

In 9–14: **a.** Find the common difference of each arithmetic sequence. **b.** Write the n th term of each sequence for the given value of n .

9. 3, 6, 9, 12, ..., $n = 8$

10. 2, 7, 12, 17, ..., $n = 12$

11. 18, 16, 14, 12, ..., $n = 10$

12. $\frac{1}{2}, 1, \frac{3}{2}, 2, \dots, n = 7$

13. $-1, -3, -5, -7, \dots, n = 10$

14. 2.1, 2.2, 2.3, 2.4, ..., $n = 20$

15. Write the first six terms of the arithmetic sequence that has 12 for the first term and 42 for the sixth term.
16. Write the first nine terms of the arithmetic sequence that has 100 as the fifth term and 80 as the ninth term.
17. Find four arithmetic means between 3 and 18.
18. Find two arithmetic means between 1 and 5.
19. Write a recursive definition for an arithmetic sequence with a common difference of -3 .

Applying Skills

20. On July 1, Mr. Taylor owed \$6,000. On the 1st of each of the following months, he repaid \$400. List the amount owed by Mr. Taylor on the 2nd of each month starting with July 2. Explain why the amount owed each month forms an arithmetic sequence.
21. Li is developing a fitness program that includes doing push-ups each day. On each day of the first week he did 20 push-ups. Each subsequent week, he increased his daily push-ups by 5. During which week did he do 60 push-ups a day?
 - a. Use a formula to find the answer to the question.
 - b. Write the arithmetic sequence to answer the question.
 - c. Which method do you think is better? Explain your answer.

22. a. Show that a linear function whose domain is the set of positive integers is an arithmetic sequence.
- b. For the linear function $y = mx + b$, $y = a_n$ and $x = n$. Express a_1 and d of the arithmetic sequence in terms of m and b .
23. Leslie noticed that the daily number of e-mail messages she received over the course of two months form an arithmetic sequence. If she received 13 messages on day 3 and 64 messages on day 20:
- a. How many messages did Leslie receive on day 12?
- b. How many messages will Leslie receive on day 50?

6-3 SIGMA NOTATION

Ken wants to get more exercise so he begins by walking for 20 minutes. Each day for two weeks, he increases the length of time that he walks by 5 minutes. At the end of two weeks, the length of time that he has walked each day is given by the following arithmetic sequence:

$$20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85$$

The total length of time that Ken has walked in two weeks is the sum of the terms of this arithmetic sequence:

$$20 + 25 + 30 + 35 + 40 + 45 + 50 + 55 + 60 + 65 + 70 + 75 + 80 + 85$$

This sum is called a *series*.

DEFINITION

A **series** is the indicated sum of the terms of a sequence.

The symbol Σ , which is the Greek letter sigma, is used to indicate a sum. The number of minutes that Ken walked on the n th day is $a_n = 20 + (n - 1)(5)$. We can write the sum of the number of minutes that Ken walked in **sigma notation**:

$$\sum_{n=1}^{14} a_n = \sum_{n=1}^{14} 20 + (n - 1)(5)$$

The “ $n = 1$ ” below Σ is the value of n for the first term of the series, and the number above Σ is the value of n for the last term of the series. The symbol

$\sum_{n=1}^{14} a_n$ can be read as “the sum of a_n for all integral values of n from 1 to 14.”

In expanded form:

$$\begin{aligned}\sum_{n=1}^{14} a_n &= a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} \\ \sum_{n=1}^{14} 20 + (n-1)(5) &= [20 + 0(5)] + [20 + 1(5)] + [20 + 2(5)] + \cdots + [20 + 13(5)] \\ &= 20 + 25 + 30 + \cdots + 85\end{aligned}$$

For example, we can indicate the sum of the first 50 positive even numbers as $\sum_{i=1}^{50} 2i$.

$$\begin{aligned}\sum_{i=1}^{50} 2i &= 2(1) + 2(2) + 2(3) + 2(4) + \cdots + 2(50) \\ &= 2 + 4 + 6 + 8 + \cdots + 100\end{aligned}$$

Note that in this case, i was used to indicate the number of the term. Although any variable can be used, n , i , and k are the variables most frequently used. Be careful not to confuse the variable i with the imaginary number i .

The series $\sum_{i=1}^{50} 2i$ is an example of a **finite series** since it is the sum of a *finite* number of terms. An **infinite series** is the sum of an *infinite* number of terms of a sequence. We indicate that a series is infinite by using the symbol for infinity, ∞ . For example, we can indicate the sum of *all* of the positive even numbers as:

$$\sum_{i=1}^{\infty} 2i = 2 + 4 + 6 + \cdots + 2i + \cdots$$

EXAMPLE 1

Write the sum given by $\sum_{k=1}^7 (k + 5)$.

Solution
$$\begin{aligned}\sum_{k=1}^7 (k + 5) &= (1 + 5) + (2 + 5) + (3 + 5) + (4 + 5) + (5 + 5) + (6 + 5) + (7 + 5) \\ &= 6 + 7 + 8 + 9 + 10 + 11 + 12 \quad \text{Answer}\end{aligned}$$

EXAMPLE 2

Write the sum of the first 25 positive odd numbers in sigma notation.

Solution The positive odd numbers are 1, 3, 5, 7,

The 1st positive odd number is 1 less than twice 1, the 2nd positive odd number is 1 less than twice 2, the 3rd positive odd number is 1 less than twice 3. In general, the n th positive odd number is 1 less than twice n or $a_n = 2n - 1$.

The sum of the first 25 odd numbers is $\sum_{n=1}^{25} (2n - 1)$. **Answer**



EXAMPLE 3

Use sigma notation to write the series $12 + 20 + 30 + 42 + 56 + 72 + 90 + 110$ in two different ways:

- Express each term as a sum of two numbers, one of which is a square.
- Express each term as a product of two numbers.

Solution a. The terms of this series can be written as $3^2 + 3, 4^2 + 4, 5^2 + 5, \dots, 10^2 + 10$, or, in general, as $n^2 + n$ with n from 3 to 10.

The series can be written as $\sum_{n=3}^{10} (n^2 + n)$.

- b. Write the series as

$$3(4) + 4(5) + 5(6) + 6(7) + 7(8) + 8(9) + 9(10) + 10(11).$$

The series is the sum of $n(n + 1)$ from $n = 3$ to $n = 10$.

The series can be written as $\sum_{n=3}^{10} n(n + 1)$.

Answers a. $\sum_{n=3}^{10} (n^2 + n)$ b. $\sum_{n=3}^{10} n(n + 1)$ ■

EXAMPLE 4

Use sigma notation to write the sum of the reciprocals of the natural numbers.

Solution The reciprocals of the natural numbers are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}$.

Since there is no largest natural number, this sequence has no last term. Therefore, the sum of the terms of this sequence is an infinite series. In sigma notation, the sum of the reciprocals of the natural numbers is:

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{Answer} \quad \text{■}$$

Finite Series and the Graphing Calculator



The graphing calculator can be used to find the sum of a finite series. We use the sum(function along with the seq(function of the previous section to evaluate

a series. For example, to evaluate $\sum_{k=1}^{37} \frac{1}{k(k+2)}$ on the calculator:

STEP 1. Enter the sequence into the calculator and store it in list L_1 .

ENTER: $\boxed{2\text{nd}} \boxed{\text{LIST}} \boxed{\blacktriangleright} \boxed{5} \boxed{1} \boxed{\div} \boxed{(} \boxed{\text{X,T,}\theta,n} \boxed{(} \boxed{\text{X,T,}\theta,n} \boxed{+} \boxed{2} \boxed{)} \boxed{)} \boxed{,} \boxed{\text{X,T,}\theta,n} \boxed{,} \boxed{1} \boxed{,} \boxed{37} \boxed{)} \boxed{\text{STO}\blacktriangleright} \boxed{2\text{nd}} \boxed{\text{L1}} \boxed{\text{ENTER}}$

DISPLAY:
$$\begin{array}{l} \text{seq}\{1/(X(X+2)),X \\ ,1,37\} \rightarrow L_1 \\ \{.3333333333 .1 \dots\} \end{array}$$

STEP 2. Use the $\text{sum}()$ function to find the sum.

ENTER: $\boxed{2\text{nd}} \boxed{\text{LIST}} \boxed{\blacktriangleright} \boxed{\blacktriangleright} \boxed{5} \boxed{2\text{nd}} \boxed{\text{L1}} \boxed{\text{ENTER}}$

DISPLAY:
$$\begin{array}{l} \text{sum}(L_1 \\ .7240215924 \end{array}$$

The sum is approximately equal to 0.72.

Exercises

Writing About Mathematics

- Is the series given in Example 3 equal to $\sum_{n=1}^8 [(n+2)^2 + (n+2)]$? Justify your answer.
- Explain why $\sum_{k=0}^{10} \frac{1}{k}$ is undefined.

Developing Skills

In 3–14: **a.** Write each arithmetic series as the sum of terms. **b.** Find each sum.

3. $\sum_{n=1}^{10} 3n$

4. $\sum_{k=1}^5 (2k - 2)$

5. $\sum_{k=1}^4 k^2$

6. $\sum_{n=1}^6 n^3$

7. $\sum_{k=1}^{10} (100 - 5k)$

8. $\sum_{n=5}^{10} (3n - 3)$

9. $\sum_{n=2}^5 (n^2 + 2i)$

10. $\sum_{h=1}^{10} (-1)^h h$

11. $\sum_{n=5}^{15} [4n - (n+1)]$

12. $\sum_{k=1}^{10} -ki$

13. $\sum_{k=3}^5 (5 - 4k)$

14. $\sum_{n=0}^5 (-2n)^{n+1}$

In 15–26, write each series in sigma notation.

15. $3 + 5 + 7 + 9 + 11 + 13 + 15$

16. $1 + 6 + 11 + 16 + 21 + 26 + 31 + 36$

17. $1^1 + 2^2 + 3^3 + 4^4 + 5^5$

18. $100 + 95 + 90 + 85 + \cdots + 5$

19. $3 + 6 + 9 + 12 + 15 + \cdots + 30$

20. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{16} + \frac{1}{32}$

21. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10}$

22. $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!}$

23. $-\frac{1}{3} + \frac{2}{9} - \frac{3}{27} + \frac{4}{81} - \frac{5}{243}$

24. $\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \frac{1}{4 \times 5} + \frac{1}{5 \times 6} + \frac{1}{6 \times 7}$

25. $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \cdots$

26. $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \cdots$

Applying Skills

27. Show that $\sum_{i=1}^n ka_i = k \sum_{i=1}^n a_i$.

28. Show that $\sum_{i=1}^n (a_i + b_i) = \sum_{i=1}^n a_i + \sum_{i=1}^n b_i$.

29. In a theater, there are 20 seats in the first row. Each row has 3 more seats than the row ahead of it. There are 35 rows in the theater.

- Express the number of seats in the n th row of the theater in terms of n .
- Use sigma notation to represent the number of seats in the theater.

30. On Monday, Elaine spent 45 minutes doing homework. On the remaining four days of the school week, she spent 15 minutes longer doing homework than she had the day before.

- Express the number of minutes Elaine spent doing homework on the n th day of the school week.
- Use sigma notation to represent the total number of minutes Elaine spent doing homework from Monday to Friday.

31. Use the graphing calculator to evaluate the following series to the nearest *hundredth*:



(1) $\sum_{n=1}^{50} \left(1 + \frac{1}{n}\right)$

(2) $\sum_{k=1}^{18} \frac{5}{1+k}$

(3) $\sum_{n=1}^{20} \frac{(n-1)(-1)^n}{n}$

6-4 ARITHMETIC SERIES

In the last section, we wrote the sequence of minutes that Ken walked each day for two weeks:

$$20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85$$

Since the difference between each pair of consecutive times is a constant, 5, the sequence is an arithmetic sequence. The total length of time that Ken walked in two weeks is the sum of the terms of this sequence:

$$20 + 25 + 30 + 35 + 40 + 45 + 50 + 55 + 60 + 65 + 70 + 75 + 80 + 85$$

In general, if $a_1, a_1 + d, a_1 + 2d, \dots, a_n + (n - 1)d$ is an arithmetic sequence with n terms, then:

$$\sum_{i=1}^n [a_1 + (i - 1)d] = a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n - 1)d]$$

This sum is called an *arithmetic series*.

DEFINITION

An **arithmetic series** is the indicated sum of the terms of an arithmetic sequence.

Once a given series is defined, we can refer to it simply as S (for sigma). S_n is called the **n th partial sum** and represents the sum of the first n terms of the sequence.

We can find the number of minutes that Ken walked in 14 days by adding the 14 numbers or by observing the pattern of this series. Begin by writing the sum first in the order given and then in reverse order.

$$S_{14} = 20 + 25 + 30 + 35 + 40 + 45 + 50 + 55 + 60 + 65 + 70 + 75 + 80 + 85$$

$$S_{14} = 85 + 80 + 75 + 70 + 65 + 60 + 55 + 50 + 45 + 40 + 35 + 30 + 25 + 20$$

Note that for this arithmetic series:

$$a_1 + a_{14} = a_2 + a_{13} = a_3 + a_{12} = a_4 + a_{11} = a_5 + a_{10} = a_6 + a_9 = a_7 + a_8$$

Add the sums together, combining corresponding terms. The sum of each pair is 105 and there are $\frac{14}{2}$ or 7 pairs.

$$S_{14} = 20 + 25 + 30 + 35 + 40 + 45 + 50 + 55 + 60 + 65 + 70 + 75 + 80 + 85$$

$$S_{14} = 85 + 80 + 75 + 70 + 65 + 60 + 55 + 50 + 45 + 40 + 35 + 30 + 25 + 20$$

$$2S_{14} = 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105 + 105$$

$$2S_{14} = 14(105) \quad \leftarrow \text{Write the expression in factored form.}$$

$$S_{14} = 7(105) \quad \leftarrow \text{Divide both sides by 2.}$$

$$S_{14} = 735 \quad \leftarrow \text{Simplify.}$$

Therefore, the total number of minutes that Ken walked in 14 days is $7(105)$ or 735 minutes.

Does a similar pattern exist for every arithmetic series? Consider the general arithmetic series with n terms, $a_n = a_1 + (n - 1)d$. List the terms of the series in ascending order from a_1 to a_n and in descending order from a_n to a_1 .

Ascending Order	Descending Order	Sum
a_1	$a_n = a_1 + (n - 1)d$	$a_1 + [a_1 + (n - 1)d] = 2a_1 + (n - 1)d$
$a_2 = a_1 + d$	$a_{n-1} = a_1 + (n - 2)d$	$[a_1 + d] + [a_1 + (n - 2)d] = 2a_1 + (n - 1)d$
$a_3 = a_1 + 2d$	$a_{n-2} = a_1 + (n - 3)d$	$[a_1 + 2d] + [a_1 + (n - 3)d] = 2a_1 + (n - 1)d$
\vdots	\vdots	\vdots
$a_{n-2} = a_1 + (n - 3)d$	$a_3 = a_1 + 2d$	$[a_1 + (n - 3)d] + [a_1 + 2d] = 2a_1 + (n - 1)d$
$a_{n-1} = a_1 + (n - 2)d$	$a_2 = a_1 + d$	$[a_1 + (n - 2)d] + [a_1 + d] = 2a_1 + (n - 1)d$
$a_n = a_1 + (n - 1)d$	a_1	$[a_1 + (n - 1)d] + a_1 = 2a_1 + (n - 1)d$

In general, for any arithmetic series with n terms there are $\frac{n}{2}$ pairs of terms whose sum is $a_1 + a_n$:

$$2S_n = n(a_1 + a_n) = n(2a_1 + (n - 1)d)$$

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n - 1)d)$$

EXAMPLE 1

- Write the sum of the first 15 terms of the arithmetic series $1 + 4 + 7 + \cdots$ in sigma notation.
- Find the sum.

Solution a. For the related arithmetic sequence, $1, 4, 7, \dots$, the common difference is 3. Therefore,

$$\begin{aligned} a_n &= 1 + (n - 1)(3) \\ &= 3n - 2 \end{aligned}$$

The series is written as $\sum_{n=1}^{15} 3n - 2$. **Answer**

- Use the formula $S_n = \frac{n}{2}(2a_1 + (n - 1)d)$ with $a_1 = 1$, $n = 15$, and $d = 3$.

$$\begin{aligned} S_{15} &= \frac{15}{2}[2(1) + (15 - 1)(3)] \\ &= \frac{15}{2}[2 + 14(3)] \\ &= \frac{15}{2}[2 + 42] \\ &= \frac{15}{2}[44] \\ &= 330 \quad \text{Answer} \end{aligned}$$

Note: Part **b** can also be solved by using the formula $S_n = \frac{n}{2}(a_1 + a_n)$. First find a_{15} .

$$\begin{aligned}
 a_{15} &= a_1 + (n - 1)d & S_{15} &= \frac{15}{2}(1 + 43) \\
 &= 1 + (15 - 1)(3) & &= \frac{15}{2}(44) \\
 &= 1 + 14(3) & &= 330 \\
 &= 1 + 42 \\
 &= 43
 \end{aligned}$$

Notice that for this series there are $7\frac{1}{2}$ pairs. The first 7 numbers are paired with the last 7 numbers, and each pair has a sum of 44. The middle number, 22, is paired with itself, making half of a pair.

EXAMPLE 2

The sum of the first and the last terms of an arithmetic sequence is 80 and the sum of all the terms is 1,200. How many terms are in the sequence?

Solution

$$\begin{aligned}
 S_n &= \frac{n}{2}(a_1 + a_n) \\
 1,200 &= \frac{n}{2}(80) \\
 1,200 &= 40n \\
 30 &= n
 \end{aligned}$$

Answer There are 30 terms in the sequence. ■

Exercises

Writing About Mathematics

1. Is there more than one arithmetic series such that the sum of the first and the last terms is 80 and the sum of the terms is 1,200? Justify your answer.
2. Is $1 + 1 + 2 + 3 + 5 + 8 + 13 + 21$ an arithmetic series? Justify your answer.

Developing Skills

In 3–8, find the sum of each series using the formula for the partial sum of an arithmetic series. Be sure to show your work.

3. $2 + 4 + 6 + 8 + 10 + 12$

4. $10 + 20 + 30 + 40 + 50 + 60$

5. $3 + 1 - 1 - 3 - 5 - 7 - 9 - 11 - 13$

6. $0i + 4i + 8i + 12i + 16i + 20i$

7. $0 + \frac{1}{3} + \frac{2}{3} + 1 + \frac{4}{3} + \frac{5}{3} + 2$

8. $\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \cdots + 15\sqrt{2}$

In 9–18, use the given information to **a.** write the series in sigma notation, and **b.** find the sum of the first n terms.

9. $a_1 = 3, a_n = 39, d = 4$

10. $a_1 = 24, a_n = 0, n = 6$

11. $a_1 = 24, a_n = 0, d = -6$

12. $a_1 = 10, d = 2, n = 14$

13. $a_1 = 2, d = \frac{1}{2}, n = 15$

14. $a_1 = 0, d = -2, n = 10$

15. $a_1 = \frac{1}{3}, d = \frac{1}{3}, n = 12$

16. $a_1 = 100, d = -5, n = 20$

17. $a_{10} = 50, d = 2.5, n = 10$

18. $a_5 = 15, d = 2, n = 12$

In 19–24: **a.** Write each arithmetic series as the sum of terms. **b.** Find the sum.

19. $\sum_{k=1}^{10} 2k$

20. $\sum_{k=2}^8 (3 + k)$

21. $\sum_{n=0}^9 (20 - 2n)$

22. $\sum_{i=0}^{19} (100 - 5i)$

23. $\sum_{n=1}^{25} -2n$

24. $\sum_{n=1}^{10} (-1 + 2n)$

Applying Skills

25. Madeline is writing a computer program for class. The first day she wrote 5 lines of code and each day, as she becomes more skilled in writing code, she writes one more line than the previous day. It takes Madeline 6 days to complete the program. How many lines of code did she write?
26. Jose is learning to cross-country ski. He began by skiing 1 mile the first day and each day he increased the distance skied by 0.2 mile until he reached his goal of 3 miles.
 - a. How many days did it take Jose to reach his goal?
 - b. How many miles did he ski from the time he began until the day he reached his goal?
27. Sarah wants to save for a special dress for the prom. The first month she saved \$15 and each of the next five months she increased the amount that she saved by \$2. What is the total amount Sarah saved over the six months?
28. In a theater, there are 20 seats in the first row. Each row has 3 more seats than the row ahead of it. There are 35 rows in the theater. Find the total number of seats in the theater.
29. On Monday, Enid spent 45 minutes doing homework. On the remaining four days of the school week she spent 15 minutes longer doing homework than she had the day before. Find the total number of minutes Enid spent doing homework from Monday to Friday.
30. Keegan started a job that paid \$20,000 a year. Each year after the first, he received a raise of \$600. What was the total amount that Keegan earned in six years?
31. A new health food store's net income was a loss of \$2,300 in its first month, but its net income increased by \$575 in each succeeding month for the next year. What is the store's net income for the year?

6-5 GEOMETRIC SEQUENCES

Pete reuses paper that is blank on one side to write phone messages. One day he took a stack of five sheets of paper and cut it into three parts and then cut each part into three parts. The number of pieces of paper that he had after each cut forms a sequence: 5, 15, 45. This sequence can be written as:

$$5, 5(3), 5(3)^2$$

Each term of the sequence is formed by multiplying the previous term by 3, or we could say that the ratio of each term to the previous term is a constant, 3. If Pete had continued to cut the pieces of paper in thirds, the terms of the sequence would be

$$5, 5(3), 5(3)^2, 5(3)^3, 5(3)^4, 5(3)^5, \dots$$

This sequence is called a *geometric sequence*.

DEFINITION

A **geometric sequence** is a sequence such that for all n , there is a constant r such that $\frac{a_n}{a_{n-1}} = r$. The constant r is called the **common ratio**.

The recursive definition of a geometric sequence is:

$$a_n = a_{n-1}r$$

When written in terms of a_1 and r , the terms of a geometric sequence are:

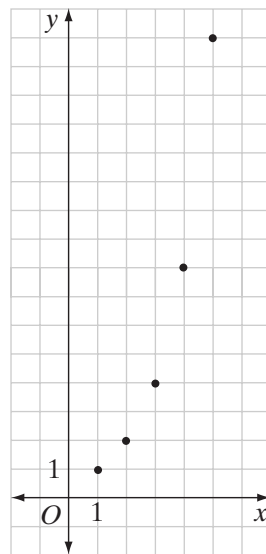
$$a_1, a_2 = a_1r, a_3 = a_1r^2, a_4 = a_1r^3, \dots$$

Each term after the first is obtained by multiplying the previous term by r . Therefore, each term is the product of a_1 times r raised to a power that is one less than the number of the term, that is:

$$a_n = a_1r^{n-1}$$

Since a sequence is a function, we can sketch the function on the coordinate plane. The geometric sequence $1, 1(2), 1(2)^2, 1(2)^3, 1(2)^4$ or $1, 2, 4, 8, 16$ can be written in function notation as $\{(1,1), (2,2), (3,4), (4,8), (5,16)\}$. Note that since the domain is the set of positive integers, the points on the graph are distinct points that are not connected by a curve.

Many common problems can be characterized by a geometric sequence. For example, if P dollars are invested at a yearly rate of 4%, then the value of the investment at the end of each year forms a geometric sequence:



$$\text{Year 1: } P + 0.04P = P(1 + 0.04) = P(1.04)$$

$$\text{Year 2: } P(1.04) + 0.04(P(1.04)) = P(1.04)(1 + 0.04) = P(1.04)(1.04) = P(1.04)^2$$

$$\text{Year 3: } P(1.04)^2 + 0.04(P(1.04)^2) = P(1.04)^2(1 + 0.04) = P(1.04)^2(1.04) = P(1.04)^3$$

$$\text{Year 4: } P(1.04)^3 + 0.04(P(1.04)^3) = P(1.04)^3(1 + 0.04) = P(1.04)^3(1.04) = P(1.04)^4$$

$$\text{Year 5: } P(1.04)^4 + 0.04(P(1.04)^4) = P(1.04)^4(1 + 0.04) = P(1.04)^4(1.04) = P(1.04)^5$$

The terms $P(1.04)$, $P(1.04)^2$, $P(1.04)^3$, $P(1.04)^4$, \dots , $P(1.04)^n$ form a geometric sequence in which $a_1 = P(1.04)$ and $r = 1.04$. The n th term is $a_n = a_1 r^{n-1} = P(1.04)(1.04)^{n-1}$.

EXAMPLE 1

Is the sequence 4, 12, 36, 108, 324, \dots a geometric sequence?

Solution In the sequence, $\frac{12}{4} = 3$, $\frac{36}{12} = 3$, $\frac{108}{36} = 3$, $\frac{324}{108} = 3$, the ratio of any term to the preceding term is a constant, 3. Therefore, 4, 12, 36, 108, 324, \dots is a geometric sequence with $a_1 = 4$ and $r = 3$. ■

EXAMPLE 2

What is the 10th term of the sequence 4, 12, 36, 108, 324, \dots ?

Solution The sequence 4, 12, 36, 108, 324, \dots is a geometric sequence with $a_1 = 4$ and $r = 3$. Therefore,

$$a_n = a_1 r^{n-1}$$

$$a_{10} = 4(3)^9$$

Use a calculator for the computation.

ENTER: 4 \times 3 $^$ 9 ENTER

DISPLAY: 4*3^9
78732

Answer 78,732 ■

Geometric Means

In the proportion $\frac{4}{20} = \frac{20}{100}$, we say that 20 is the mean proportional between 4 and 100. These three numbers form a geometric sequence 4, 20, 100. The mean proportional, 20, is also called the **geometric mean** between 4 and 100.

Between two numbers, there can be any number of geometric means. For example, to write three geometric means between 4 and 100, we want to form a geometric sequence 4, a_2 , a_3 , a_4 , 100. In this sequence, $a_1 = 4$ and $a_5 = 100$.

$$\begin{aligned}a_n &= a_1 r^{n-1} \\a_5 &= a_1 r^4 \\100 &= 4r^4 \\25 &= r^4 \\\sqrt[4]{25} &= \sqrt[4]{r^4} \\r &= \pm \sqrt{5}\end{aligned}$$

There are two possible values of r , $-\sqrt{5}$ and $+\sqrt{5}$. Therefore, there are two possible sequences and two possible sets of geometric means.

One sequence is 4, $4\sqrt{5}$, 20, $20\sqrt{5}$, 100 with $4\sqrt{5}$, 20, and $20\sqrt{5}$ three geometric means between 4 and 100.

The other sequence is 4, $-4\sqrt{5}$, 20, $-20\sqrt{5}$, 100 with $-4\sqrt{5}$, 20, and $-20\sqrt{5}$ three geometric means between 4 and 100.

Note that for each sequence, the ratio of each term to the preceding term is a constant.

$\frac{4\sqrt{5}}{4} = \sqrt{5}$	$\frac{-4\sqrt{5}}{4} = -\sqrt{5}$
$\frac{20}{4\sqrt{5}} = \frac{5}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}$	$\frac{20}{-4\sqrt{5}} = \frac{5}{-\sqrt{5}} = \frac{5\sqrt{5}}{-5} = -\sqrt{5}$
$\frac{20\sqrt{5}}{20} = \sqrt{5}$	$\frac{-20\sqrt{5}}{20} = -\sqrt{5}$
$\frac{100}{20\sqrt{5}} = \frac{5}{\sqrt{5}} = \frac{5\sqrt{5}}{5} = \sqrt{5}$	$\frac{100}{-20\sqrt{5}} = \frac{5}{-\sqrt{5}} = \frac{5\sqrt{5}}{-5} = -\sqrt{5}$

EXAMPLE 3

Find four geometric means between 5 and 1,215.

Solution We want to find the missing terms in the sequence 5, a_2 , a_3 , a_4 , a_5 , 1,215. Use the formula to determine the common ratio r for $a_1 = 5$ and $a_6 = 1,215$.

$$\begin{aligned}a_n &= a_1 r^{n-1} \\a_6 &= a_1 r^5 \\1,215 &= 5r^5 \\243 &= r^5 \\\sqrt[5]{243} &= \sqrt[5]{r^5} \\r &= 3\end{aligned}$$

The four geometric means are $5(3) = 15$, $15(3) = 45$, $45(3) = 135$, and $135(3) = 405$.

Answer 15, 45, 135, and 405

Exercises

Writing About Mathematics

1. Autumn said that the answer to Example 2 could have been found by entering 4 \times 3 ENTER on a calculator and then entering \times 3 ENTER eight times to display the sequence to the 9 terms after the first. Do you think that this is an easier way to find the 10th term? Explain your answer.
2. Sierra said that $8, 8\sqrt{2}, 16, 16\sqrt{2}, 32$ is a geometric sequence with three geometric means, $8\sqrt{2}, 16$, and $16\sqrt{2}$. Do you agree with Sierra? Justify your answer.

Developing Skills

In 3–14, determine whether each given sequence is geometric. If it is geometric, find r . If it is not geometric, explain why it is not.

- | | |
|---|--|
| 3. $4, 8, 16, 32, 64, \dots$ | 4. $1, 5, 25, 125, 625, \dots$ |
| 5. $3, 6, 9, 12, \dots$ | 6. $\frac{1}{2}, 2, 8, 32, \dots$ |
| 7. $1, -3, 9, -27, 81, \dots$ | 8. $36, 12, 4, \frac{4}{3}, \dots$ |
| 9. $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \dots$ | 10. $\frac{5}{2}, 2, \frac{3}{2}, 1, \frac{1}{2}, \dots$ |
| 11. $1, -10, 100, -1,000, 10,000, \dots$ | 12. $1, 0.1, 0.01, 0.001, 0.0001, \dots$ |
| 13. $0.05, -0.1, 0.2, -0.4, \dots$ | 14. a, a^2, a^3, a^4, \dots |

In 15–26, write the first five terms of each geometric sequence.

- | | | |
|---------------------------------|---------------------------------|--------------------------|
| 15. $a_1 = 1, r = 6$ | 16. $a_1 = 40, r = \frac{1}{2}$ | 17. $a_1 = 2, r = 3$ |
| 18. $a_1 = \frac{1}{4}, r = -2$ | 19. $a_1 = 1, r = \sqrt{2}$ | 20. $a_1 = 10, a_2 = 30$ |
| 21. $a_1 = -1, a_2 = 4$ | 22. $a_1 = 100, a_3 = 1$ | 23. $a_1 = 1, a_3 = 16$ |
| 24. $a_1 = 1, a_3 = 2$ | 25. $a_1 = 1, a_4 = -8$ | 26. $a_1 = 81, a_5 = 1$ |
27. What is the 10th term of the geometric sequence $0.25, 0.5, 1, \dots$?
 28. What is the 9th term of the geometric sequence $125, 25, 5, \dots$?
 29. In a geometric sequence, $a_1 = 1$ and $a_5 = 16$. Find a_9 .
 30. The first term of a geometric sequence is 1 and the 4th term is 27. What is the 8th term?

- 31. In a geometric sequence, $a_1 = 2$ and $a_3 = 16$. Find a_6 .
- 32. In a geometric sequence, $a_3 = 1$ and $a_7 = 9$. Find a_1 .
- 33. Find two geometric means between 6 and 93.75.
- 34. Find three geometric means between 3 and $9\frac{13}{27}$.
- 35. Find three geometric means between 8 and 2,592.

Applying Skills

- 36. If \$1,000 was invested at 6% annual interest at the beginning of 2001, list the geometric sequence that is the value of the investment at the beginning of each year from 2001 to 2010.
- 37. Al invested \$3,000 in a certificate of deposit that pays 5% interest per year. What is the value of the investment at the end of each of the first four years?
- 38. In a small town, a census is taken at the beginning of each year. The census showed that there were 5,000 people living in the town at the beginning of 2001 and that the population decreased by 2% each year for the next seven years. List the geometric sequence that gives the population of the town from 2001 to 2008. (A decrease of 2% means that the population changed each year by a factor of 0.98.) Write your answer to the nearest integer.
- 39. It is estimated that the deer population in a park was increasing by 10% each year. If there were 50 deer in the park at the end of the first year in which a study was made, what is the estimated deer population for each of the next five years? Write your answer to the nearest integer.
- 40. A car that cost \$20,000 depreciated by 20% each year. Find the value of the car at the end of each of the first four years. (A depreciation of 20% means that the value of the car each year was 0.80 times the value the previous year.)
- 41. A manufacturing company purchases a machine for \$50,000. Each year the company estimates the depreciation to be 15%. What will be the estimated value of the machine after each of the first six years?

6-6 GEOMETRIC SERIES

DEFINITION

A **geometric series** is the indicated sum of the terms of a geometric sequence.

For example, 3, 12, 48, 192, 768, 3,072 is a geometric sequence with $r = 4$. The indicated sum of this sequence, $3 + 12 + 48 + 192 + 768 + 3,072$, is a geometric series.

In general, if $a_1, a_1r, a_1r^2, a_1r^3, \dots, a_1r^{n-1}$ is a geometric sequence with n terms, then

$$\sum_{i=1}^n a_1 r^{i-1} = a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1}$$

is a geometric series.

Let the sum of these six terms be S_6 . Now multiply S_6 by the *negative* of the common ratio, -4 . This will result in a series in which each term but the last is the *opposite* of a term in S_6 .

$$\begin{array}{rcl} S_6 & = & 3 + 12 + 48 + 192 + 3,072 \\ -4S_6 & = & -12 - 48 - 192 - 3,072 - 12,288 \\ \hline -3S_6 & = & 3 - 12,288 & \leftarrow \text{Add the sums.} \\ S_6 & = & \frac{3 - 12,288}{-3} & \leftarrow \text{Divide by } -3. \\ & = & -1 + 4,096 = 4,095 & \leftarrow \text{Simplify.} \end{array}$$

Thus, $S_6 = 4,095$. The pattern of a geometric series allows us to find a formula for the sum of the series. To S_n , add $-rS_n$:

$$\begin{array}{rcl} S_n & = & a_1 + a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-1} \\ -rS_n & = & -a_1r - a_1r^2 - a_1r^3 - \cdots - a_1r^{n-1} - a_1r^n \\ \hline S_n - rS_n & = & a_1 - a_1r^n \\ S_n(1 - r) & = & a_1(1 - r^n) \\ S_n & = & \frac{a_1(1 - r^n)}{1 - r} \end{array}$$

EXAMPLE 1

Find the sum of the first 10 terms of the geometric series $2 + 1 + \frac{1}{2} + \cdots$.

Solution For the series $2 + 1 + \frac{1}{2} + \cdots$, $a_1 = 2$ and $r = \frac{1}{2} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$.

$$\begin{aligned} S_n &= \frac{a_1(1 - r^n)}{1 - r} \\ S_{10} &= \frac{2\left(1 - \left(\frac{1}{2}\right)^{10}\right)}{1 - \frac{1}{2}} = \frac{2\left(1 - \frac{1}{1,024}\right)}{\frac{1}{2}} = \frac{2\left(\frac{1,023}{1,024}\right)}{\frac{1}{2}} = 2\left(\frac{1,023}{1,024}\right) \times \frac{2}{1} = \frac{1,023}{256} \end{aligned}$$

Answer $S_{10} = \frac{1,023}{256}$



EXAMPLE 2

Find the sum of five terms of the geometric series whose first term is 2 and whose fifth term is 162.

Solution Use $a_5 = a_1 r^{5-1}$ to find r .

$$162 = 2r^4$$

$$81 = r^4$$

$$\sqrt[4]{81} = \sqrt[4]{r^4}$$

$$\pm 3 = r$$

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_5 = \frac{2(1 - 3^5)}{1 - 3} = \frac{2(1 - 243)}{-2} = \frac{2(-242)}{-2} = 242$$

$$S_5 = \frac{2(1 - (-3)^5)}{1 - (-3)} = \frac{2(1 + 243)}{4} = \frac{2(244)}{4} = 122$$

Answer $S_5 = 242$ or $S_5 = 122$

Exercises**Writing About Mathematics**

- Casey said that the formula for the sum of a geometric series could be written as $S_n = \frac{a_1 - a_n r}{1 - r}$. Do you agree with Casey? Justify your answer.
- Sherri said that Example 1 could have been solved by simply adding the ten terms of the series on a calculator. Do you think that this would have been a simpler way of finding the sum? Explain why or why not.

Developing Skills

In 3–14, find the sum of n terms of each geometric series.

3. $a_1 = 1, r = 2, n = 12$

4. $a_1 = 4, r = 3, n = 11$

5. $a_2 = 6, r = 4, n = 15$

6. $a_1 = 10, r = 10, n = 6$

7. $a_3 = 0.4, r = 2, n = 12$

8. $a_1 = 1, r = \frac{1}{3}, n = 10$

9. $5 + 10 + 20 + \cdots + a_n, n = 8$

10. $1 + 5 + 25 + \cdots + a_n, n = 10$

11. $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + a_n, n = 6$

12. $2 - 8 + 32 - \cdots + a_n, n = 7$

13. $a_1 = 4, a_5 = 324, n = 9$

14. $a_1 = 1, a_8 = 128, n = 10$

In 15–22: **a.** Write each sum as a series. **b.** Find the sum of each series.

15. $3 + \sum_{n=1}^5 3(2)^n$

16. $1 + \sum_{n=1}^5 \left(\frac{1}{3}\right)^n$

17. $10 + \sum_{n=1}^5 10\left(\frac{1}{2}\right)^n$

18. $-6 + \sum_{n=1}^8 -6(4)^n$

19. $1 + \sum_{n=1}^5 (-2)^n$

20. $1 + \sum_{n=1}^5 \left(\frac{2}{3}\right)^n$

21. $100 + \sum_{k=1}^6 100\left(\frac{1}{2}\right)^k$

22. $-81 + \sum_{k=1}^6 -81\left(-\frac{1}{3}\right)^k$

23. Find the sum of the first six terms of the series $1 + \sqrt{3} + 3 + 3\sqrt{3} + \dots$

24. Find the sum of the first eight terms of a series whose first term is 1 and whose eighth term is 625.

Applying Skills

25. A group of students are participating in a math contest. Students receive 1 point for their first correct answer, 2 points for their second correct answer, 4 points for their third correct answer, and so forth. What is the score of a student who answers 10 questions correctly?
26. Heidi deposited \$400 at the beginning of each year for six years in an account that paid 5% interest. At the end of the sixth year, her first deposit had earned interest for six years and was worth $400(1.05)^6$ dollars, her second deposit had earned interest for five years and was worth $400(1.05)^5$ dollars, her third deposit had earned interest for four years and was worth $400(1.05)^4$ dollars. This pattern continues.
- What is the value of Heidi's sixth deposit at the end of the sixth year? Express your answer as a product and as a dollar value.
 - Do the values of these deposits after six years form a geometric sequence? Justify your answer.
 - What is the total value of Heidi's six deposits at the end of the sixth year?
27. A ball is thrown upward so that it reaches a height of 9 feet and then falls to the ground. When it hits the ground, it bounces to $\frac{1}{3}$ of its previous height. If the ball continues in this way, bouncing each time to $\frac{1}{3}$ of its previous height until it comes to rest when it hits the ground for the fifth time, find the total distance the ball has traveled, starting from its highest point.
28. If you start a job for which you are paid \$1 the first day, \$2 the second day, \$4 the third day, and so on, how many days will it take you to become a millionaire?

6-7 INFINITE SERIES

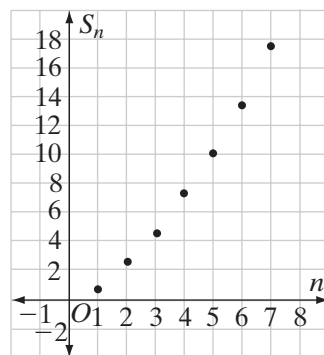
Recall that an infinite series is a series that continues without end. That is, for a series $S_n = a_1 + a_2 + \dots + a_n$, we say that n approaches infinity. We can also write $\sum_{n=1}^{\infty} a_n$. However, while there may be an infinite number of terms, a series can behave in only one of three ways. The series may increase without limit, decrease without limit, or approach a limit.

CASE 1 *The series increases without limit.*

Consider an arithmetic series $S_n = \frac{n}{2}(2a_1 + (n-1)d)$.
If $a_1 = 1$ and $d = \frac{1}{2}$:

$$\begin{aligned} S_n &= \frac{n}{2} \left[2(1) + (n-1)\left(\frac{1}{2}\right) \right] \\ &= \frac{n}{2} \left[\frac{4}{2} + \frac{n}{2} - \frac{1}{2} \right] \\ &= \frac{n}{2} \left[\frac{3+n}{2} \right] = \frac{3n+n^2}{4} \end{aligned}$$

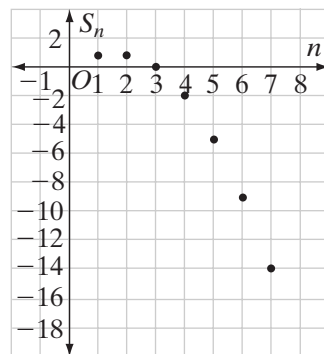
As the value of n approaches infinity, $\frac{3n+n^2}{4}$ increases without limit. This series has no limit. We can see this by graphing the function $S_n = \frac{3n+n^2}{4}$ for positive integer values.

**CASE 2** *The series decreases without limit.*

For an arithmetic series $S_n = \frac{n}{2}(2a_1 + (n-1)d)$, if $a_1 = 1$ and $d = -1$:

$$\begin{aligned} S_n &= \frac{n}{2} [2(1) + (n-1)(-1)] \\ &= \frac{n}{2} [2 - n + 1] = \frac{n}{2} [3 - n] \end{aligned}$$

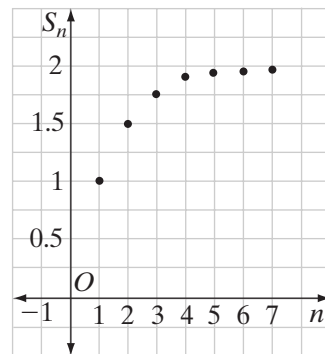
For this series, as the value of n approaches infinity, $\frac{n}{2}[3-n]$ or $\frac{3n-n^2}{2}$ decreases without limit. This series also has no limit. We can see this by graphing the function $S_n = \frac{3n-n^2}{2}$ for positive integer values.

**CASE 3** *The series approaches a limit.*

For a geometric series $S_n = \frac{a_1 - a_1 r^n}{1 - r}$, if $a_1 = 1$ and $r = \frac{1}{2}$:

$$\begin{aligned} S_n &= \frac{1 - 1\left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}} \\ &= 2 - 2\left(\frac{1}{2}\right)^n = 2 - \left(\frac{1}{2}\right)^{n-1} \end{aligned}$$

As n approaches infinity, $\left(\frac{1}{2}\right)^{n-1}$ approaches 0. Therefore, $S_n = 2 - \left(\frac{1}{2}\right)^{n-1}$ approaches $S_n = 2 - 0 = 2$ as n approaches infinity.



This series has a limit. The diagram on the bottom of page 274 shows how the sum of n terms approaches 2 as the number of terms, n , increases.

In general,

- ▶ **An infinite arithmetic series has no limit.**
- ▶ **An infinite geometric series has no limit when $|r| > 1$.**
- ▶ **An infinite geometric series has a finite limit when $|r| < 1$.**

When $|r| < 1$, r^n approaches 0. Therefore:

- ▶ **$S_n = \frac{a_1 - a_1 r^n}{1 - r}$ approaches $\frac{a_1}{1 - r}$ as n approaches infinity.**

EXAMPLE I

Find $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$.

Solution

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \cdots$$

This is an infinite geometric series with $a_1 = \frac{1}{3}$ and $r = \frac{1}{3}$.

Using the formula $\frac{a_1 - a_1 r^n}{1 - r}$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}}$$

As n approaches infinity, $\left(\frac{1}{3}\right)^n$ approaches 0. Therefore,

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{3} \times \frac{3}{2} = \frac{1}{2} \quad \text{Answer}$$

Series Containing Factorials

Series that are neither arithmetic nor geometric must be considered individually. Often the limit of a series can be found to be finite by comparing the terms of the series to the terms of another series with a known limit. One such series

is $\sum_{k=1}^n \frac{1}{k!}$.

Recall *factorials* from previous courses. We write **n factorial** as follows:

DEFINITION

$$n! = n(n-1)(n-2)(n-3) \cdots (3)(2)(1)$$

Consider the series $\sum_{k=1}^n \frac{1}{k!} = \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$. This series can be shown to have a limit by comparing its terms to the terms of the geometric series with $a_1 = 1$ and $r = \frac{1}{2}$. Each term, a_n ($n > 1$), of this geometric series is $\frac{1}{2^{n-1}}$.

n	Term of Geometric Series	Term of Series to Be Tested	Comparison of Terms
1	1	$\frac{1}{1!} = 1$	$1 = 1$
2	$\frac{1}{2^1} = \frac{1}{2}$	$\frac{1}{2!} = \frac{1}{2 \times 1} = \frac{1}{2}$	$\frac{1}{2} = \frac{1}{2}$
3	$\frac{1}{2^2} = \frac{1}{4}$	$\frac{1}{3!} = \frac{1}{3 \times 2 \times 1} = \frac{1}{6}$	$\frac{1}{4} > \frac{1}{6}$
4	$\frac{1}{2^3} = \frac{1}{8}$	$\frac{1}{4!} = \frac{1}{4 \times 3 \times 2 \times 1} = \frac{1}{24}$	$\frac{1}{8} > \frac{1}{24}$
5	$\frac{1}{2^4} = \frac{1}{16}$	$\frac{1}{5!} = \frac{1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{120}$	$\frac{1}{16} > \frac{1}{120}$

We can conclude that $\frac{1}{n!} = \frac{1}{n(n-1)(n-2)\cdots(2)(1)} \leq \frac{1}{2^{n-1}}$. Therefore:

$$\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}$$

Previously, we found that for the geometric series with $a_1 = 1$ and $r = \frac{1}{2}$, the series approaches $\frac{1}{1 - \frac{1}{2}}$ or 2 as n approaches infinity. Therefore, $\sum_{n=1}^{\infty} \frac{1}{n!} < 2$. Thus, this series is bounded above by 2.

To find a lower bound, notice that the series $\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is the sum of positive terms. Therefore, for $n > 3$,

$$\begin{aligned} \frac{1}{1!} + \frac{1}{2!} &< \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \\ \text{or } \frac{3}{2} &< \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \end{aligned}$$

Putting it all together, what we have shown is that for the series $\sum_{n=1}^{\infty} \frac{1}{n!}$:

$$\frac{3}{2} < \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 2$$

$$\text{or } \frac{3}{2} < \sum_{n=1}^{\infty} \frac{1}{n!} < 2$$

The Number e

Let us add 1 to both sides of the inequality derived in the previous section. We find that as n approaches infinity,

$$1 + \frac{3}{2} < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 1 + 2$$

or $\frac{5}{2} < 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} < 3$

The infinite series $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$ is equal to an irrational number that is greater than $\frac{5}{2}$ and less than 3. We call this number e .

Eighteenth-century mathematicians computed this number to many decimal places and assigned to it the symbol e just as earlier mathematicians computed to many decimal places the ratio of the length of the circumference of a circle to the length of the diameter and assigned the symbol π to this ratio. Therefore, we say:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} + \cdots = e$$



This number has an important role in many different branches of mathematics. A calculator will give the value of e to nine decimal places.

ENTER: **2nd** **e^x** 1 **ENTER**

DISPLAY: $e^{\{1\}}$ 2.718281828

EXAMPLE 2

Find, to the nearest hundredth, the value of $1 + e^2$.

Solution Evaluate the expression on a calculator.

ENTER: 1 **+** **2nd** **e^x** 2 **ENTER**

DISPLAY: $1+e^{\{2\}}$ 8.389056099

Answer To the nearest hundredth, $1 + e^2 \approx 8.39$.

Exercises

Writing About Mathematics

1. Show that if the first term of an infinite geometric series is 1 and the common ratio is $\frac{1}{c}$, then the sum is $\frac{c}{c-1}$.
2. Cody said that since the calculator gives the value of e as 2.718281828, the value of e can be written as $2.\overline{71828}$, a repeating decimal and therefore a rational number. Do you agree with Cody? Explain why or why not.

Developing Skills

In 3–10: **a.** Write each series in sigma notation. **b.** Determine whether each sum increases without limit, decreases without limit, or approaches a finite limit. If the series has a finite limit, find that limit.

3. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$

4. $2 + \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$

5. $2 + 4 + 6 + 8 + \cdots$

6. $5 + 1 + \frac{1}{5} + \frac{1}{25} + \cdots$

7. $5 + 1 - 3 - 7 - \cdots$

8. $6 + 3 + \frac{3}{2} + \frac{3}{4} + \cdots$

9. $\frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n+1)!} + \cdots$

10. $1 + 3 + 6 + 10 + \cdots$

In 11–16, an infinitely repeating decimal is an infinite geometric series. Find the rational number represented by each of the following infinitely repeating decimals.

11. $1.11111 \dots = 1 + 0.1 + 0.01 + 0.001 + \cdots$

12. $0.33333 \dots = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots$

13. $0.444444 \dots$

14. $0.121212 \dots = 0.12 + 0.0012 + 0.000012 + \cdots$

15. $0.242424 \dots$

16. $0.126126126 \dots$

17. The sum of the infinite series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n}$ is 2. Find values of n such that

$$2 - a_n < \frac{2^{26} - 1}{2^{25}}.$$

CHAPTER SUMMARY

A **sequence** is a set of numbers written in a given order. Each term of a sequence is associated with the positive integer that specifies its position in the ordered set. A **finite sequence** is a function whose domain is the set of integers $\{1, 2, 3, \dots, n\}$. An **infinite sequence** is a function whose domain is the set of positive integers. The terms of a sequence are often designated as a_1, a_2, a_3, \dots . The formula that allows any term of a sequence except the first to be computed from the previous term is called a **recursive definition**.

An **arithmetic sequence** is a sequence such that for all n , there is a constant d such that $a_{n+1} - a_n = d$. For an arithmetic sequence:

$$a_n = a_1 + (n - 1)d = a_{n-1} + d$$

A **geometric sequence** is a sequence such that for all n , there is a constant r such that $\frac{a_{n+1}}{a_n} = r$. For an arithmetic sequence:

$$a_n = a_1 r^{n-1} = a_{n-1} r$$

A **series** is the indicated sum of the terms of a sequence. The Greek letter Σ is used to indicate a sum defined for a set of consecutive integer.

If S_n represents the **n th partial sum**, the sum of the first n terms of a sequence, then

$$S_n = \sum_{k=1}^n a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

For an arithmetic series:

$$S_n = \frac{n}{2}(a_1 + a_n) = \frac{n}{2}(2a_1 + (n - 1)d)$$

For a geometric series:

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

For a geometric series, if $|r| < 1$ and n approaches infinity:

$$S_n = \frac{a_1}{1 - r} \quad \text{or} \quad \sum_{n=1}^{\infty} a_1 r^{n-1} = \frac{a_1}{1 - r}$$

As n approaches infinity:

$$\sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{(n-1)!} = e$$

The number e is an irrational number.

VOCABULARY

6-1 Sequence • Finite sequence • Infinite sequence • Recursive definition

6-2 Common difference • Arithmetic sequence • Arithmetic means

6-3 Series • Σ • Sigma notation • Finite series • Infinite series • ∞

6-4 Arithmetic series • S_n • n th partial sum

6-5 Geometric sequence • Common ratio • Geometric mean

6-6 Geometric series

6-7 n factorial • e

REVIEW EXERCISES

In 1–6: **a.** Write a recursive formula for a_n . **b.** Is the sequence arithmetic, geometric, or neither? **c.** If the sequence is arithmetic or geometric, write an explicit formula for a_n . **d.** Find a_{10} .

1. $1, 5, 9, 13, \dots$

2. $3, 1, \frac{1}{3}, \frac{1}{9}, \dots$

3. $12, 11, 10, 9, \dots$

4. $1, 3, 6, 10, 15, \dots$

5. $1i, 3i, 7i, 15i, 31i, \dots$

6. $2, -6, 18, -54, 162, \dots$

In 7–12, write each series in sigma notation.

7. $1 + 3 + 6 + 10 + 15 + 21 + 28 + 36$

8. $2 + 5 + 8 + 11 + 14 + 17 + 20$

9. $4 + 6 + 8 + 10 + 12 + 14$

10. $1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2$

11. $1 - 2 + 3 - 4 + 5 - 6 + 7$

12. $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \dots$

13. In an arithmetic sequence, $a_1 = 6$ and $d = 5$. Write the first five terms.

14. In an arithmetic sequence, $a_1 = 6$ and $d = 5$. Find a_{30} .

15. In an arithmetic sequence, $a_1 = 5$ and $a_4 = 23$. Find a_{12} .

16. In an arithmetic sequence, $a_3 = 0$ and $a_{10} = 70$. Find a_1 .

17. In a geometric sequence, $a_1 = 2$ and $a_2 = 5$. Write the first five terms.

18. In a geometric sequence, $a_1 = 2$ and $a_2 = 5$. Find a_{10} .

19. In a geometric sequence, $a_1 = 2$ and $a_4 = 128$. Find a_6 .

20. Write a recursive formula for the sequence $12, 20, 30, 42, 56, 70, 90, 110, \dots$

21. Write the first five terms of a sequence if $a_1 = 1$ and $a_n = 4a_{n-1} + 3$.
22. Find six arithmetic means between 1 and 36.
23. Find three geometric means between 1 and 625.
24. In an arithmetic sequence, $a_1 = 1$, $d = 8$. If $a_n = 89$, find n .
25. In a geometric sequence, $a_1 = 1$ and $a_5 = \frac{1}{4}$. Find r .
26. Write $\sum_{n=1}^4 3^n$ as the sum of terms and find the sum.
27. Write $\sum_{k=0}^6 (12 - 3k)$ as the sum of terms and find the sum.
28. To the nearest hundredth, find the value of $3e^3$.
29. a. Write, in sigma notation, the series $3 + 1.5 + 0.75 + 0.375 + 0.1875 + \cdots$.
b. Determine whether the series given in part **a** increases without limit, decreases without limit, or approaches a finite limit. If the series has a finite limit, find that limit.
30. A grocer makes a display of canned tomatoes that are on sale. There are 24 cans in the first (bottom) layer and, after the first, each layer contains three fewer cans than in the layer below. There are three cans in the top layer.
a. How many layers of cans are in the display?
b. If the grocer sold all of the cans in the display, how many cans of tomatoes did he sell?
31. A retail store pays cashiers \$20,000 a year for the first year of employment and increases the salary by \$500 each year.
a. What is the salary of a cashier in the tenth year of employment?
b. What is the total amount that a cashier earns in his or her first 10 years?
32. Ben started a job that paid \$40,000 a year. Each year after the first, his salary was increased by 4%.
a. What was Ben's salary in his eighth year of employment?
b. What is the total amount that Ben earned in eight years?
33. The number of handshakes that are exchanged if every person in a room shakes hands once with every other person can be written as a sequence. Let a_{n-1} be the number of handshakes exchanged when there are $n - 1$ persons in the room. If one more person enters the room so that there are n persons in the room, that person shakes hands with $n - 1$ persons, creating $n - 1$ additional handshakes.
a. Write a recursive formula for a_n .
b. If $a_1 = 0$, write the first 10 terms of the sequence.
c. Write a formula for a_n in terms of n .



Exploration

The graphing calculator can be used to graph recursive sequences and examine the *convergence*, *divergence*, or *oscillation* of the sequence. A sequence **converges** if as n goes to infinity, the sequence approaches a limit. A sequence **diverges** if as n goes to infinity, the sequence increases or decreases without limit. A sequence **oscillates** if as n goes to infinity, the sequence neither converges nor diverges.

For example, to verify that the geometric sequence $a_n = a_{n-1}\left(\frac{1}{2}\right)$ with $a_1 = 5$ converges to 0:

STEP 1. Set the calculator to sequence graphing mode. Press **MODE** and select Seq, the last option in the fourth row.

```
NORMAL SCI ENG
FLOAT 0123456789
RADIAN DEGREE
FUNC PAR POL Seq
CONNECTED DOT
SEQUENTIAL SIMUL
REAL A+Bi Re^θI
FULL HORIZ G-T
```

STEP 2. Enter the expression $a_{n-1}\left(\frac{1}{2}\right)$ as $u(n-1)\left(\frac{1}{2}\right)$ into $u(n)$. The function name u is found above the **7** key.

ENTER: **Y=** **2nd** **u** **(** **X,T,θ,n**
- **1** **)** **×** **1** **÷** **2**.

```
PLOT1 PLOT2 PLOT3
nMin=1
u(n)=u(n-1)*1/2
u(nMin)=5
v(n)=
v(nMin)=
w(n)=
```

STEP 3. To set the initial value of $a_1 = 5$, set $u(nMin) = 5$.

ENTER: **▼** **5**

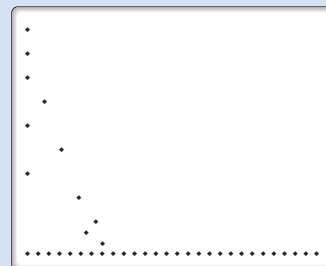
(To set more than one initial value, for example $a_1 = 5$ and $a_2 = 6$, set $u(nMin) = \{5, 6\}$.)

STEP 4. Graph the sequence up to $n = 25$.

Press **WINDOW** **▼** **25** to set $nMax = 25$.

```
WINDOW
nMin=1
nMax=25
PLOTSTART=1
PLOTSTEP=1
Xmin=-10
Xmax=10
↓ Xscl=1
```

STEP 5. Finally, press **ZOOM** **0** to graph the sequence. You can use **TRACE** to explore the values of the sequence. We can see that the sequence approaches the value of 0 very quickly.



In 1–4: **a.** Graph each sequence up to the first 75 terms. **b.** Use the graph to make a conjecture as to whether the sequence converges, diverges, or oscillates.

1. $a_1 = 1, a_2 = 1, a_n = a_{n-2} + a_{n-1}$ (Fibonacci sequence)

2. $a_1 = 5, a_n = \frac{1}{a_{n-1}}$

3. $a_1 = 100, a_n = \frac{1}{2}a_{n-1} + 100$

4. $a_1 = 10, a_n = 0.95a_{n-1} + 500(0.75)^n$

CUMULATIVE REVIEW

CHAPTERS 1–6

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If the domain is the set of integers, then the solution set of $-2 \leq x + 3 < 7$ is

(1) $\{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}$

(2) $\{-2, -1, 0, 1, 2, 3, 4, 5, 6\}$

(3) $\{-5, -4, -3, -2, -1, 0, 1, 2, 3, 4\}$

(4) $\{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$

2. The expression $(a + 3)^2 - (a + 3)$ is equal to

(1) $a^2 - a + 6$

(3) $a^2 + 5a + 12$

(2) $a^2 + 5a + 6$

(4) $a + 3$

3. In simplest form, $\frac{x}{3} + \frac{x-2}{4} - \frac{x-2}{8}$ is equal to

(1) $\frac{11x-6}{24}$

(3) $\frac{11x+6}{24}$

(2) $\frac{11x-18}{24}$

(4) $\frac{x(x-2)^2}{24}$

4. The sum of $(3 - \sqrt{5}) + (2 - \sqrt{45})$ is
(1) $5 - 4\sqrt{5}$ (2) $5 - 3\sqrt{5}$ (3) $5 - 5\sqrt{2}$ (4) $\sqrt{2}$
5. The fraction $\frac{1}{1 - \sqrt{2}}$ is equal to
(1) $1 + \sqrt{2}$ (2) $-1 + \sqrt{2}$ (3) $-1 - \sqrt{2}$ (4) $1 - \sqrt{2}$
6. The quadratic equation $3x^2 - 7x = 3$ has roots that are
(1) real, rational, and equal.
(2) real, rational, and unequal.
(3) real irrational and unequal.
(4) imaginary.
7. A function that is one-to-one is
(1) $y = 2x + 1$ (3) $y = x^3 + x$
(2) $y = x^2 + 1$ (4) $y = |x|$
8. The function $y = x^2$ is translated 2 units up and 3 units to the left. The equation of the new function is
(1) $y = (x - 3)^2 + 2$ (3) $y = (x + 3)^2 + 2$
(2) $y = (x - 3)^2 - 2$ (4) $y = (x + 3)^2 - 2$
9. Which of the following is a real number?
(1) i (2) i^2 (3) i^3 (4) $2i$
10. The sum of the roots of a quadratic equation is -4 and the product of the roots is 5 . The equation could be
(1) $x^2 - 4x + 5 = 0$ (3) $x^2 - 4x - 5 = 0$
(2) $x^2 + 4x - 5 = 0$ (4) $x^2 + 4x + 5 = 0$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicated the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work will receive only 1 credit.

11. What number must be added to each side of the equation $x^2 + 3x = 0$ in order to make the left member the perfect square of a binomial?
12. Solve for x and check: $7 - \sqrt{x + 2} = 4$.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicated the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work will receive only 1 credit.

- 13.** Express the fraction $\frac{(2 + i)^2}{i}$ in $a + bi$ form.
- 14.** Graph the solution set of the inequality $2x^2 - 5x - 3 > 0$.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicated the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work will receive only 1 credit.

- 15. a.** Write the function that is the inverse function of $y = 3x + 1$.
- b.** Sketch the graph of the function $y = 3x + 1$ and of its inverse.
- c.** What are the coordinates of the point of intersection of the function and its inverse?
- 16. a.** Write a recursive definition for the geometric sequence of five terms in which $a_1 = 3$ and $a_5 = 30,000$.
- b.** Write the sum of the sequence in part **a** in sigma notation.
- c.** Find the sum from part **b**.